

# BIRATIONAL MORI FIBER STRUCTURES OF $\mathbb{Q}$ -FANO 3-FOLD WEIGHTED COMPLETE INTERSECTIONS, III

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ABSTRACT. This is a continuation of a series of papers studying the birational Mori fiber structures of anticanonically embedded  $\mathbb{Q}$ -Fano 3-fold weighted complete intersections of codimension 2. We have proved that 19 families consists of birationally rigid varieties and 14 families consists of birationally birigid varieties. The aim of this paper is to continue the work systematically and prove that, among the remaining 52 families, 21 families consist of birationally birigid varieties.

## 1. INTRODUCTION

This is a continuation of a series of papers studying birational Mori fiber structures of anticanonically embedded  $\mathbb{Q}$ -Fano 3-fold weighted complete intersections of codimension 2 ( $\mathbb{Q}$ -Fano WCIs of codimension 2, for short). Here, a Mori fiber space that is birational to a given algebraic variety is called a *birational Mori fiber structure* of the variety. There are 85 families of  $\mathbb{Q}$ -Fano 3-fold WCIs of codimension 2 (see [7]). In [12], we divide 85 families into the disjoint union of 3 pieaces  $I := \{1, 2, \dots, 85\} = I_{br} \cup I_{dP} \cup I_F$ , where  $|I_{br}| = 19$ ,  $|I_{dP}| = 6$  and  $|I_F| = 60$ , and studied their birational (non-)rigidity. We proved birational rigidity of general members of family No.  $i$  with  $i \in I_{br}$ , and among other things, we also proved that a member of family No.  $i$  with  $i \in I_{dP}$  and  $i \in I_F$  is birational to a del Pezzo fiber space over  $\mathbb{P}^1$  and to a  $\mathbb{Q}$ -Fano 3-fold, respectively. Recently, Ahmadinezhad and Zucconi [2] succeeded in removing generality conditions for family No.  $i$  with  $i \in I_{br}$  and proved birational rigidity of every quasismooth member. From now on, we focus on families No.  $i$  with  $i \in I_F$  and recall the following result for such families.

**Theorem 1.1** ([12, Theorem 1.3]). *Let  $X$  be a general member of family No.  $i$  with  $i \in I_F$ . Then there exists a  $\mathbb{Q}$ -Fano 3-fold weighted hypersurface  $X'$  that is birational but not isomorphic to  $X$ . Moreover, each maximal singularity on  $X$  is untwisted by a Sarkisov link that is either a birational involution or a link to  $X'$ .*

This is an important step toward the determination of the birational Mori fiber structures of  $X$ . In fact, by continuing the work, we proved in [13] birational birigidity of members of family No.  $i$  with  $i \in I_{cA/x} \cup I_{cAx/4}$  for suitable subsets  $I_{cAx/2}$  and  $I_{cAx/4}$  of  $I_F$ . Here, *birational birigidity* means that there exist exactly two Mori fiber spaces in the birational equivalence class.

Note that, in Theorem 1.1, the weighted hypersurface  $X'$  is uniquely determined by  $X$  and we call it the *birational counterpart* of  $X$ . It has a unique non-quotient terminal singular point together with some other terminal quotient singular points. We define subsets  $I_{cA/n}$  and  $I_{cD/3}$  of  $I_F$  by the following rule: we have  $i \in I_{cA/n}$  (resp.  $i \in I_{cD/3}$ ) if and only if the non-quotient terminal singular point of the birational

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counterpart of a general member  $X$  of family No.  $i$  is of type  $cA/n$  (resp.  $cD/3$ ). Here, we regard  $cA/1$  as  $cA$ . Specifically, those subsets are give as

$$I_{cA/n} = \{3, 6, 7, 9, 10, 16, 18, 21, 22, 26, 28, 33, 36, 38, 44, 48, 52, 57, 63\},$$

$$I_{cD/3} = \{61, 62\}.$$

Family No. 3 has been studied in this direction: Corti and Mella [4] proved birational birigidity of a general  $\mathbb{Q}$ -Fano WCI  $X = X_{3,4} \subset \mathbb{P}(1, 1, 1, 1, 2, 2)$  of cubic and quartic. Note that  $3 \in I_{cA/n} \subset I_F$ . The aim of this paper is to complete the determination of Mori fiber structures of families No.  $i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ , where  $I_{cA/n}^* := I_{cA/n} \setminus \{3\}$ . We state the main theorem and its direct consequence.

**Theorem 1.2.** *Let  $X$  be a general member of family No.  $i$  with  $i \in I_{cA/n} \cup I_{cD/3}$ . Then  $X$  is birationally birigid. More precisely,  $X$  is birational to a  $\mathbb{Q}$ -Fano 3-fold weighted hypersurface  $X'$  and is not birational to any other Mori fiber space.*

**Corollary 1.3.** *A general member of family No.  $i$  is not rational for  $i \in I_{cA/n} \cup I_{cD/3}$ .*

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## 2. PRELIMINARIES

**2.1. Notation and convention.** Throughout the paper, we work over the field  $\mathbb{C}$  of complex numbers. A normal projective variety  $X$  is said to be a  $\mathbb{Q}$ -Fano variety if  $-K_X$  is ample, it is  $\mathbb{Q}$ -factorial, has only terminal singularities and its Picard number is 1. We say that an algebraic fiber space  $X \rightarrow S$  is a *Mori fiber space* if  $X$  is a normal projective  $\mathbb{Q}$ -factorial variety with at most terminal singularities,  $\dim S < X$ ,  $-K_X$  is relatively ample over  $S$  and the relative Picard number is 1.

Let  $X$  be a normal projective  $\mathbb{Q}$ -factorial variety,  $\mathcal{H}$  a linear system on  $X$ ,  $D \subset X$  a Weil divisor and  $C \subset X$  a curve. We say that  $\mathcal{H}$  is  $\mathbb{Q}$ -linearly equivalent to  $D$ , denoted by  $\mathcal{H} \sim_{\mathbb{Q}} D$ , if a member of  $\mathcal{H}$  is  $\mathbb{Q}$ -linearly equivalent to  $D$ . We define  $(\mathcal{H} \cdot C) := (H \cdot C)$  for  $H \in \mathcal{H}$ . In this paper, a *divisorial extraction* (or *divisorial contraction*) is always an extremal divisorial extraction (or contraction) in the Mori category.

A weighted projective space (WPS)  $\mathbb{P}(a, b, c, d, e)$  with homogeneous coordinates  $x, y, z, s, t$  is sometimes denoted by  $\mathbb{P}(a_x, b_y, c_z, d_s, e_t)$ . A closed subscheme  $Z$  in  $\mathbb{P}(a_0, \dots, a_n)$  is *quasismooth* (resp. *quasismooth outside a point p*) if the affine cone  $C_Z \subset \mathbb{A}^{n+1}$  is smooth outside the origin (resp. outside the closure of the inverse image of  $\mathbf{p}$  via the morphism  $\mathbb{A}^{n+1} \setminus \{o\} \rightarrow \mathbb{P}(a_0, \dots, a_n)$ ). For  $i = 0, 1, \dots, n$ , we denote by  $\mathbf{p}_i$  the vertex  $(0 : \dots : 1 : \dots : 0)$  of  $\mathbb{P}(a_0, \dots, a_n)$ , where the 1 is in the  $(i + 1)$ -th position.

The main object of this paper is a weighted hypersurface  $X'$  birational to a member  $X$  of family No.  $i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ . Let  $\mathbb{P} := \mathbb{P}(a_0, \dots, a_4)$  be the ambient WPS of  $X'$ . We write  $x_0, \dots, x_3, w$  (resp.  $x, y, z, t, \dots, w$ ) for the homogeneous coordinates when we treat several families at a time (resp. a specific family). For example, we write  $x_0, x_1, y, z, w$  (resp.  $x, y, z, t, w$ ) for the coordinates of  $\mathbb{P}(1, 1, 2, 3, 2)$  (resp.  $\mathbb{P}(1, 2, 3, 5, 4)$ ). The coordinate  $w$  is distinguished so that the vertex at which only the coordinate  $w$  is non-zero is the unique  $cA/n$  or  $cD/3$  point of  $X'$ . For homogeneous polynomials  $f_1, \dots, f_m$  in the variables  $x_0, \dots, x_3, w$  or  $x, y, z, \dots, w$ , we denote by  $(f_1 = \dots = f_m = 0)$  the closed subscheme of  $\mathbb{P}$  defined by the homogeneous ideal  $(f_1, \dots, f_m)$  and denote by  $(f_1 = \dots = f_m = 0)_{X'}$  the scheme-theoretic

intersection  $(f_1 = \cdots = f_m = 0) \cap X'$ . For a polynomial  $f = f(x_0, \dots, x_3, w)$  and a monomial  $x_0^{c_0} \cdots x_3^{c_3} w^d$ , we write  $x_0^{c_0} \cdots x_3^{c_3} w^d \in f$  (resp.  $x_0^{c_0} \cdots x_3^{c_3} w^d \notin f$ ) if the coefficient of the monomial in  $f$  is non-zero (resp. zero).

A *weighted complete intersection curve* (*WCI curve*) of type  $(c_1, c_2, c_3)$  (resp. of type  $(c_1, c_2, c_3, c_4)$ ) in  $\mathbb{P}(a_0, \dots, a_4)$  (resp.  $\mathbb{P}(a_0, \dots, a_5)$ ) is an irreducible and reducible curve defined by three (resp. four) homogeneous polynomials of degree  $c_1, c_2$  and  $c_3$  (resp.  $c_1, c_2, c_3$  and  $c_4$ ).

**2.2. Weighted blowup.** We fix notation on weighted blowups of cyclic quotients of affine spaces.

Let  $\mathbb{A} := \mathbb{A}^n$  be the affine  $n$ -space with affine coordinates  $x_1, \dots, x_n$  and let  $b_1, \dots, b_n$  be positive integers. Let  $\Phi: \mathbb{A} \dashrightarrow \mathbb{P} := \mathbb{P}(b_1, \dots, b_n)$  the rational map defined by  $(\alpha_1, \dots, \alpha_n) \mapsto (\alpha_1 : \cdots : \alpha_n)$  and  $\text{Wbl}(\mathbb{A}) \subset \mathbb{A} \times \mathbb{P}$  the graph of  $\Phi$ . Suppose that we are given a  $\mathbb{Z}_r$ -action on  $\mathbb{A}$  of type  $\frac{1}{r}(a_1, \dots, a_n)$ . We assume that  $b_i \equiv a_i \pmod{r}$  for every  $i$ . Let  $V$  be the quotient of  $\mathbb{A}$  by the  $\mathbb{Z}_r$ -action and  $X$  a subvariety of  $V$  through the origin. The rational map  $\Phi$  descends to a rational map  $\Phi_V: V \dashrightarrow \mathbb{P}$ . We define  $\text{Wbl}(V)$  (resp.  $\text{Wbl}(X)$ ) to be the graph of  $\Phi_V$  (resp.  $\Phi_V|_X$ ) and call the projection  $\varphi_V: \text{Wbl}(V) \rightarrow V$  (resp.  $\varphi_X: \text{Wbl}(X) \rightarrow X$ ) the *weighted blowup* of  $V$  (resp.  $X$ ) at the origin with  $\text{wt}(x_1, \dots, x_n) = \frac{1}{r}(b_1, \dots, b_n)$ . This weight is referred to as the  $\varphi_V$ -weight or the  $\varphi_X$ -weight. We consider the  $\mathbb{Z}_r$ -action on  $\mathbb{A} \times \mathbb{P}$  which acts on  $\mathbb{A}$  as above and on  $\mathbb{P}$  trivially. We have a natural morphism  $\text{Wbl}(\mathbb{A}) \rightarrow \text{Wbl}(V)$  and from this we can see  $\text{Wbl}(V)$  as the quotient of  $\text{Wbl}(\mathbb{A})$  by the  $\mathbb{Z}_r$ -action.

We explain orbifold charts of  $\text{Wbl}(V)$  and  $\text{Wbl}(X)$ . Let  $X_1, \dots, X_n$  be the homogeneous coordinates of  $\mathbb{P}$ . For each  $i$ , we define  $\text{Wbl}_i(\mathbb{A})$  (resp.  $\text{Wbl}_i(V)$ ) to be the open subset of  $\text{Wbl}(\mathbb{A})$  (resp.  $\text{Wbl}(V)$ ) which is the intersection of  $\text{Wbl}(\mathbb{A})$  (resp.  $\text{Wbl}(V)$ ) and the open subset  $\mathbb{A} \times (X_i \neq 0)$  (resp.  $V \times (X_i \neq 0)$ ). Let  $U_i(\mathbb{A})$  be an affine  $n$ -space with affine coordinates  $\tilde{x}_1, \dots, \tilde{x}_{i-1}, x'_i, \tilde{x}_{i+1}, \dots, \tilde{x}_n$  and define a morphism  $U_i(\mathbb{A}) \rightarrow \mathbb{A}$  by the identification  $x_i = x'_i$  and  $x_j = \tilde{x}_j x'_i{}^{b_j}$  for  $j \neq i$ . We consider the  $\mathbb{Z}_{b_i}$ -action on  $U_i(\mathbb{A})$  of type  $\frac{1}{b_i}(b_1, \dots, b_{i-1}, -1, b_{i+1}, \dots, b_n)$ . We see that the quotient  $U_i(\mathbb{A})/\mathbb{Z}_{b_i}$  is naturally isomorphic to  $\text{Wbl}_i(\mathbb{A})$  and the section  $x'_i$  cuts out the open subset  $(X_i \neq 0)$  of the exceptional divisor  $\mathbb{P}$ . Note that  $\text{Wbl}_i(V)$  is the quotient of  $\text{Wbl}_i(\mathbb{A})$  by the  $\mathbb{Z}_r$ -action. The  $\mathbb{Z}_r$ -action on  $\text{Wbl}_i(\mathbb{A})$  is given by  $x'_i \mapsto \zeta_r x'_i$  and  $\tilde{x}_j \mapsto \tilde{x}_j$  for  $j \neq i$ . Let  $U_i(V)$  be the affine  $n$ -space with affine coordinates  $\tilde{x}_1, \dots, \tilde{x}_n$ . The identification  $\tilde{x}_i = x'_i{}^r$  defines a morphism  $U_i(\mathbb{A}) \rightarrow U_i(V)$  and we see that  $\text{Wbl}_i(V)$  is the quotient of  $U_i(V)$  by the  $\mathbb{Z}_{b_i}$ -action on  $U_i(V)$  of type  $\frac{1}{b_i}(b_1, \dots, b_{i-1}, -r, b_{i+1}, \dots, b_n)$ . We call  $U_i(V)$  the *orbifold chart* of  $\text{Wbl}_i(V)$  and call  $\tilde{x}_1, \dots, \tilde{x}_n$  *orbifold coordinates* of  $\text{Wbl}_i(V)$ .

We keep the above setting and let  $X$  be a closed subvariety of  $V$  which is a complete intersection defined by  $\mathbb{Z}_r$ -semi-invariant polynomials  $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$ . Let  $m_i/r$  be the order of  $f_i$  with respect to the  $\varphi_X$ -weight and we define  $g_i$  (resp.  $h_i$ ) to be the weight =  $m_i/r$  part (resp. the weight =  $m_i/r + 1$  part) of  $f_i$ . We define

$$f_j^{(i)} := f(\tilde{x}_1, \dots, \tilde{x}_i^{1/r}, \dots, \tilde{x}_n) / \tilde{x}_i^{m_i} \in \mathbb{C}[\tilde{x}_1, \dots, \tilde{x}_n]$$

and

$$U_i(X) := (f_1^{(i)} = \cdots = f_k^{(i)} = 0) \subset U_i(V).$$

Then, the quotient  $\text{Wbl}_i(X) := U_i(X)/\mathbb{Z}_{b_i}$  is an open subset of  $\text{Wbl}(X)$ . We call  $U_i(X)$  the *orbifold chart* of  $\text{Wbl}(X)$ . Assume that  $(g_1 = \cdots = g_k = 0) \subset \mathbb{A}^n$  is a

local complete intersection at the origin. Then, we have an isomorphism

$$E \cong (g_1 = g_2 = \cdots = g_k = 0) \subset \mathbb{P}(b_1, b_2, \dots, b_n),$$

where  $E$  is the exceptional divisor of the weighted blowup and, by a slight abuse of notation,  $\mathbb{P}(b_1, \dots, b_n)$  is a weighted projective space with coordinates  $x_1, \dots, x_n$ . We denote by  $J_{C_E}$  the Jacobian matrix of the affine cone  $C_E$  of  $E$ . Note that

$$J_{C_E} = \left( \frac{\partial g_i}{\partial x_j} \right)_{1 \leq i \leq k, 1 \leq j \leq n}$$

is a  $k \times n$  matrix. We define the matrix  $J_\varphi$  to be the  $k \times (n+1)$  matrix

$$J_\varphi := (J_{C_E} \ h),$$

where  $h := {}^t(h_1 \ h_2 \ \cdots \ h_k)$ .

**Lemma 2.1.** *We keep the above setting. If  $J_\psi$  is of rank  $k$  at every point of  $E$ , then  $\text{Wbl}(X)$  has at most cyclic quotient singular points and*

$$\text{Sing}(\text{Wbl}(X)) \cap E = \text{Sing}(E) \cap \text{Sing}(\mathbb{P}(b_1, \dots, b_n)).$$

*Proof.* Let  $J_{U_i(X)}$  be the Jacobian matrix of the orbifold chart  $U_i(X) \subset \mathbb{A}^n$ . Let  $\tilde{E}_i$  be the inverse image of  $E \cap \text{Wbl}_i(X)$  by  $U_i(X) \rightarrow \text{Wbl}_i(X)$ . For each  $\mathbf{q} \in \tilde{E}_i$ , we have  $\text{rank } J_{U_i(X)}(\mathbf{q}) = \text{rank } J_\varphi(\mathbf{q})$ , hence  $\text{rank } J_{U_i(X)} = k$  along  $\tilde{E}_i$ . It follows that  $U_i(X)$  is nonsingular along  $\tilde{E}_i$  for each  $i$ . This shows that  $\text{Wbl}(X)$  has at most cyclic quotient singular points. Since  $U_i(X)$  is nonsingular, the singularities of  $\text{Wbl}(X)$  come from the actions by cyclic groups. The rest is immediate from this observation.  $\square$

**2.3. Generality conditions and the definition of families.** The following condition is introduced in [12].

**Condition 2.2.** Let  $X$  be a member of family No.  $i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ .

- (C<sub>0</sub>)  $X$  is quasismooth.
- (C<sub>1</sub>) The monomial in Table 1 appears in one of the defining polynomial of  $X$  with non-zero coefficient.
- (C<sub>2</sub>)  $X$  does not contain any WCI curve listed in Table 2.
- (C<sub>2</sub>) If  $i = 21$ , then  $(x_0 = x_1 = 0)_X$  is an irreducible curve.
- (C<sub>4</sub>) If  $i = 6$  (resp. 9), then  $X$  satisfies [12, Lemma 7.3] (resp. [12, Lemma 7.11]).

TABLE 1. Monomials

No.	Monomial	No.	Monomial
9	$yz$	33	$zs$
22	$zs$	48	$zs$
28	$zs$	57	$st$

Let  $X$  be a member of family No.  $i$  satisfying Condition 2.2. Then, by [12], there is a Sarkisov link to an anticanonically embedded  $\mathbb{Q}$ -Fano 3-fold weighted hypersurface  $X'$ . Note that  $X'$  is uniquely determined by  $X$  and we call  $X'$  the *birational counterpart* of  $X$ . Precise descriptions of birational counterparts will be given in the next subsection.

TABLE 2. Type of WCI curves

No.	Type	No.	Type
6	(1,1,1,2)	33	(1,1,5,6)
10	(1,1,2,3)	38	(1,3,4,7)
16	(1,1,2,3), (1,1,3,4), (1,1,3,6)	44	(1,2,3,5)
18	(1,1,2,3)	52	(1,3,5,8)
22	(1,1,4,5)	57	(1,2,7,9)
26	(1,1,3,4)	63	(1,3,5,8)

If  $i \in I_{cA}^* \setminus \{7, 10\}$  (resp.  $i \in I_{cD/3}$ ), then the birational counterpart  $X'$  of any member  $X$  of family No.  $i$  satisfying Condition 2.2 has a singularity of type  $cA/n$  (resp.  $cD/3$ ) at  $\mathbf{p}_4$ . This is not the case for families No. 7 and 10. We introduce the following additional generality condition which ensures that the unique non-quotient singular points of birational counterparts are of type  $cA/n$ .

**Condition 2.3.** If  $i = 7$  (resp. 10), then  $\partial F_1/\partial z_0$  is not proportional to  $\partial F_1/\partial z_1$ , that is, there is no constant  $\alpha$  such that  $\partial F_1/\partial z_0 = \alpha \partial F_1/\partial z_1$ , where  $z_0, z_1$  are the coordinates of degree 3 and  $F_1$  is the defining polynomial of degree 4 (resp. 5).

**Definition 2.4.** For  $i \in I_{cA/n}^* \cup I_{cD/3}$ , we define  $\tilde{\mathcal{G}}_i$  to be the subfamily of family No.  $i$  consisting of members satisfying Conditions 2.2 and 2.3. We define  $\tilde{\mathcal{G}}'_i$  to be the family of birational counterparts of members of  $\tilde{\mathcal{G}}_i$ .

We introduce further generality conditions for suitable families in the next subsection.

**2.4. Standard defining polynomials and additional conditions.** Let  $X = X_{d_1, d_2} \subset \mathbb{P}(a_0, \dots, a_5)$  be a member of  $\tilde{\mathcal{G}}_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ . Let  $x_0, \dots, x_5$  be the homogeneous coordinates of the ambient space and let  $F_1, F_2$  be the defining polynomials of degree respectively  $d_1, d_2$ . We assume  $d_1 \leq d_2$ .

**Lemma 2.5.** Suppose  $i \in I_{cA/n}^*$ . After re-ordering and replacing coordinates,  $X$  is defined in  $\mathbb{P}(1, n, a_2, a_3, a_2 + n, a_3 + n)$  by the polynomials of the form

$$\begin{aligned} F_1 &= x_5x_2 + x_4x_3 + f(x_0, x_1, x_2, x_3), \\ F_2 &= x_5x_4 - g(x_0, x_1, x_2, x_3). \end{aligned}$$

*Proof.* We assume  $a_5 \geq a_j$  for every  $j$ . If  $X \in \tilde{\mathcal{G}}_i$  with  $i \notin \{7, 10\}$ , then  $X$  passes through  $\mathbf{p}_5$ . If  $X \in \mathcal{G}_i$  with  $i \in \{7, 10\}$ , we can assume that  $X$  passes through  $\mathbf{p}_5$  after replacing coordinates. We see that there are  $0 \leq j_1 < j_2 \leq 3$  such that  $a_5 + a_{j_1} = d_1$  and  $a_5 + a_{j_2} = d_2$ . After re-ordering coordinates, we assume that  $i_1 = 2$  and  $i_2 = 4$ . Then, by quasismoothness of  $X$  at  $\mathbf{p}_5$ , we have  $x_5x_2 \in F_1$  and  $x_5x_4 \in F_2$ . After replacing  $x_2$  and  $x_4$ , we can write  $F_1 = x_5x_2 + G_1$  and  $F_2 = x_5x_4 + G_2$  for some  $G_1, G_2 \in \mathbb{C}[x_0, \dots, x_4]$ . Moreover, by filtering off terms divisible by  $x_4$  in  $F_2$  and then replacing  $x_5$ , we assume that  $G_2$  does not involve  $x_4$ .

Suppose that  $i \in \{7, 10, 16, 18, 21, 26, 36, 38, 44, 52, 63\}$ . In this case  $d_1 \leq 2a_4$ . We claim that, after replacing coordinates other than  $x_2, x_4, x_5$ , we may assume  $x_4x_3 \in F_1$  and  $x_4^2 \notin F_1$ . Suppose that  $i \in \{21, 36, 38, 52, 63\}$ . Then  $d_1 < 2a_4$  and hence there is no monomial divisible by  $x_4^2$  in  $F_1$ . It follows that  $\mathbf{p}_4 \in X$  and there is a unique  $j$  such that  $x_4x_j \in F_1$  since  $X$  is quasismooth. By setting  $j = 3$ , we have

$x_3x_4 \in F_1$  and  $x_4^2 \notin F_1$ . Suppose that  $i \in \{7, 10\}$ . In this case, we have  $d_1 < 2a_4$  and  $a_4 = a_5$ . We can write  $F_1 = x_5x_2 + x_4f_1 + f_2$  for some  $f_1, f_2 \in \mathbb{C}[x_0, x_1, x_2, x_3]$ . Note that  $\deg f_1 = a_2$  and there is at least one  $j \neq 2$  such that  $a_j = a_2$ . If  $f_j$  does not involve coordinates other than  $x_2$ , then  $\partial F_1/\partial x_5$  and  $\partial F_1/\partial x_4$  are proportional. This is impossible by (C<sub>5</sub>). It follows that there is  $j \neq 2$  such that  $x_j \in f_1$ . By setting  $j = 3$ , we have  $x_4x_3 \in F_1$  and  $x_4^2 \notin F_1$ . If  $i \in \{16, 18, 26, 44\}$ , then  $d_1 = 2a_4$  and in this case there is  $j \neq 4$  such that  $a_j = a_4$ . We may assume that  $j = 3$ . By quasismoothness of  $X$ , the polynomial  $F_1(0, 0, 0, x_3, x_4, 0)$  cannot be a square. In particular, at least one of  $x_3x_4$  and  $x_3^2$  appear in  $F_1$  with non-zero coefficient. After replacing  $x_3$ , we may assume that  $x_3x_4 \in F_1$  and  $x_4^2 \notin F_1$ . Therefore, in any case, we can write  $F_1 = x_5x_2 + x_4x_3 + x_4f_1 + f_2$  and  $F_2 = x_5x_4 + G_2$ , where  $f_1, f_2, G_2 \in \mathbb{C}[x_0, x_1, x_2, x_3]$ . After replacing  $x_3 \mapsto x_3 - f_1$ , we obtain the desired defining polynomials.

Suppose that  $i \in \{9, 22, 28, 33, 48, 57\}$ . In this case  $2a_4 < d_1 < 3a_4$ . Then there is a unique  $j \neq 2, 4, 5$  such that  $2a_j = d_2$ . We assume  $j = 3$ . By Condition (C<sub>1</sub>), we have  $x_4x_3 \in F_1$ . Then, since  $d_1 < 3a_4$  and  $d_2 = 2a_3$ , we can write  $F_1 = x_5x_2 + x_4x_3 + x_4^2f_1 + x_4f_2 + f_3$  and  $F_2 = x_5x_4 + x_3^3 + x_3g_1 + g_2$ , where  $f_1, f_2, f_3 \in \mathbb{C}[x_0, x_1, x_2, x_3]$  and  $g_1, g_2 \in \mathbb{C}[x_0, x_1]$ . Then by the replacement,

$$x_3 \mapsto x_3 - x_4f_1 + x_2f_1^2 - f_2, x_5 \mapsto x_5 - x_4f_1^2 + 2x_3f_1 + f_1g_1,$$

we can eliminate terms divisible by  $x_4$  in  $F_1$  and  $F_2$  (other than  $x_4x_3$  in  $F_1$ ). Thus,  $F_1$  and  $F_2$  are in the desired forms.

Finally, we observe that  $\{a_0, a_1\} = \{1, n\}$  and  $a_5 - a_3 = a_4 - a_2 = n$ . Thus by interchanging  $x_0$  and  $x_1$  if necessary, we may assume that  $a_0 = 1, a_1 = n, a_4 = a_2 + n$  and  $a_5 = a_3 + n$ . This completes the proof.  $\square$

**Lemma 2.6.** *Suppose that  $X \in \tilde{\mathcal{G}}_i$ ,  $i \in I_{cA/n}^*$ , is defined by  $F_1, F_2$  in Lemma 2.5. Then the birational counterpart  $X' \in \tilde{\mathcal{G}}'_i$  is the weighted hypersurface defined by the polynomial*

$$F' = w^2x_2x_3 + wf + g$$

in  $\mathbb{P}(1, n, a_2, a_3, n)$ , where  $w$  is the homogeneous coordinates of degree  $n$  other than  $x_1$ .

*Proof.* It is proved in [12, Section 4.2] that if a member  $X = X_{d_1, d_2} \subset \mathbb{P}(a_0, \dots, a_5)$  of  $\mathcal{G}_i$ , where  $a_5 \geq a_i$  for  $i = 0, 1, 2, 3, 4$  and  $d_1 > d_2$ , is defined by polynomials  $F_1 = x_5x_2 + G_1$  and  $F_2 = x_5x_4 + G_2$ , where  $G_1, G_2 \in \mathbb{C}[x_0, \dots, x_4]$ , then the birational counterpart  $X'$  is the weighted hypersurface in  $\mathbb{P}(a_0, a_1, a_2, a_3, a_4 - a_2)$  with homogeneous coordinates  $x_0, \dots, x_3, w$  defined by

$$wG_1(x_0, x_1, x_2, x_3, wx_2) - G_2(x_0, x_1, x_2, x_3, wx_2) = 0.$$

This proves the lemma.  $\square$

**Lemma 2.7.** (1) *After replacing homogeneous coordinates, defining polynomials of  $X \in \mathcal{G}_{61}$  can be written as*

$$\begin{aligned} F_1 &= ux_0 + s^3 + s^2f_4 + sf_8 + f_{12}, \\ F_2 &= us - g_{15}, \end{aligned}$$

where  $x_0, x_1, s, y, z, u$  are the homogeneous coordinates of  $\mathbb{P}(1, 1, 4, 5, 6, 11)$  and  $f_j, g_{15} \in \mathbb{C}[x_0, x_1, y, z]$ .

- (2) After replacing homogeneous coordinates, defining polynomials of  $X \in \mathcal{G}_{62}$  can be written as

$$\begin{aligned} F_1 &= uy + s^2 + sf_6 + f_{12}, \\ F_2 &= us - g_{15}, \end{aligned}$$

where  $x, y, z, t, s, u$  are the homogeneous coordinates of  $\mathbb{P}(1, 3, 4, 5, 6, 9)$  and  $f_j, g_{15} \in \mathbb{C}[x, y, z, t]$ .

*Proof.* This is straightforward and we omit the proof.  $\square$

**Lemma 2.8.** Suppose that  $X \in \mathcal{G}_{61}$  (resp.  $\mathcal{G}_{62}$ ) is defined by  $F_1, F_2$  in Lemma 2.7. Then the birational counterpart  $X' \in \mathcal{G}'_{61}$  (resp.  $\mathcal{G}'_{62}$ ) is the weighted hypersurface defined by the polynomial

$$\begin{aligned} F' &:= w^4 x_0^3 + w^3 x_0^2 f_4 + w^2 x_0 f_8 + w f_{12} + g_{15}, \\ (\text{resp. } F' &:= w^3 y^2 + w^2 y f_6 + w f_{12} + g_{15}) \end{aligned}$$

in  $\mathbb{P}(1_{x_0}, 1_{x_1}, 5_y, 6_z, 3_w)$  (resp.  $\mathbb{P}(1_x, 3_y, 4_z, 5_t, 3_w)$ ).

*Proof.* This is proved by the same argument as in that of Lemma 2.6.  $\square$

**Definition 2.9.** Let  $X' \in \tilde{\mathcal{G}}'_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ . A defining polynomial given in Lemmas 2.6 and 2.8 is called a *standard defining polynomial* of  $X'$ .

**Remark 2.10.** In the big table, a standard defining polynomial of each family is given. For some families, specific monomials are given right after the polynomial. This is a condition imposed on members of  $\tilde{\mathcal{G}}'_i$  which is a consequence of conditions (C<sub>2</sub>) and (C<sub>3</sub>) for families other than No. 7 (see Example 2.11). For family No. 7, the condition  $y^2 \in f_4$  will be imposed in Condition 2.12 below.

**Example 2.11.** Let  $X' = X'_5 \subset \mathbb{P}(1, 1, 1, 2, 1)$  be a member of  $\tilde{\mathcal{G}}'_6$  with standard defining polynomial  $F' = w^2 x_0 y + w f_4 + g_5$ . The birational counterpart  $X \subset \mathbb{P}(1, 1, 1, 2, 2, 3)$  is defined by  $F_1 = z x_0 + y_0 y_1 + f_4(x_0, x_1, x_2, y_1)$  and  $F_2 = z y_0 - g_5(x_0, x_1, x_2, y_1)$ . If  $y^2 \notin f_4$ , then  $X$  contains the WCI curve  $(x_0 = x_1 = x_2 = y_0 = 0)$  of type  $(1, 1, 1, 2)$ . This is impossible by (C<sub>2</sub>). Thus  $y^2 \in f_4$ .

We give an another example. Let  $X' = X'_9 \subset \mathbb{P}(1, 1, 2, 3, 3)$  be a member of  $\tilde{\mathcal{G}}'_{21}$  with standard defining polynomial  $F' = w^2 x_0 y + w f_6 + g_9$ . The birational counterpart  $X \in \tilde{\mathcal{G}}_{21}$  is defined by  $t x_0 + s y + f_6$  and  $F_2 = t s - g_9$  in  $\mathbb{P}(1_{x_0}, 1_{x_1}, 2_y, 3_z, 4_s, 5_t)$ . We can write  $F_1(0, 0, y, z, s, t) = s y + \alpha z^2 + \beta y^3$  and  $F_2(0, 0, y, z, s, t) = t s - (\gamma z^3 + \delta z y^3)$ . If  $\alpha = 0$  or  $\beta = 0$ , then  $(x_0 = x_1 = 0)_X$  is reducible, which is impossible by (C<sub>3</sub>). Hence  $z^2 \in f_6$  and  $y^3 \in f_6$ . By the same reason, the case  $\gamma = \delta = 0$  cannot happen. This implies that if  $z^3 \notin g_9$ , then  $z y^3 \in g_9$ .

We introduce additional conditions on members of  $\tilde{\mathcal{G}}'_i$  for  $i \in \{6, 7, 9, 10, 16, 18, 21\}$ . We explain that a member  $X'$  of family  $\tilde{\mathcal{G}}'_i$  listed in Table 3 is defined by the polynomial  $F'$  given in the second column of Table 3. For families No. 10, 16 and 21,  $F'$  is a standard defining polynomial. Let  $X' \in \tilde{\mathcal{G}}'_6$ . Then a standard defining polynomial of  $X'$  is of the form  $F' = w^2 x_0 y + w f_4 + g_5$ , where  $f_4, g_5 \in \mathbb{C}[x_0, x_1, x_2, y]$ . Since  $y^2 \in f_4$  by generality condition, we can write  $F' = w^2 x_0 y + w(y^2 + y a_2 + a_4) + y^2 b_1 + y b_3 + b_5$ , where  $a_j, b_j \in \mathbb{C}[x_0, x_1, x_2]$ , after re-scaling  $y$ . Then, after replacing  $w \mapsto w - b_1$ , we may assume  $b_1 = 0$ . This is the one given in the second column. Similarly, defining polynomials of members of families No. 9 and 18 are given as in the second column. Note that, for family No. 7, the assertion  $y^2 \in f_4$  does not follow from Condition 2.2 and it is imposed in the following condition.



**Condition 2.12.** For a member  $X$  of family  $\tilde{\mathcal{G}}_i$  listed in Table 3, the defining polynomial of the birational counterpart  $X'$  of  $X$  is of the form in the second column and each system of equations in the third column do not have a non-trivial solution.

TABLE 3. System of equations

No.	$F'$	Equations
6	$w^2x_0y + w(y^2 + ya_2 + a_4) + yb_3 + b_5$	$x_0 = a_2 = a_4 = 0$ $x_0 = b_3 = b_5 = 0$ $x_0 = a_4 = b_5 - a_2 \frac{\partial a_4}{\partial x_0} = 0$
7	$w^2x_0x_1 + w(y^2 + ya_2 + a_4) + yb_4 + b_6$	$x_0 = b_4 = b_6 = 0$ $x_1 = b_4 = b_6 = 0$ $x_0 = x_1 = 4a_4 - a_2^2 = 0$
9	$w^2x_0y + w(ya_2 + a_6) + y^2 + yb_3 + b_6$	$x_0 = a_2 = a_5 = 0$
10	$w^2y_0y_1 + wf_5 + g_6$	$y_0 = y_1 = f_5 = g_6 = 0$
16	$w^2yz + wf_6 + g_7$	$y = f_6 = g_7 = f_6(x_0, x_1, 0, 0) = 0$
18	$w^2x_0z + wf_6 + zb_5 + b_8$	$x_0 = f_6 = zb_5 + b_8 = \frac{\partial f_6}{\partial z} = 0$ $x_0 = b_5 = b_8 = 0$
21	$w^2x_0y + wf_6 + g_9 = 0$	$x_0 = f_6 = g_9 = \frac{\partial f_6}{\partial f_6} = 0.$ $y = f_6 = g_9 = \frac{\partial f_6}{\partial z} = 0$

**Definition 2.13.** For  $i \in I_{cA/n}^* \cup I_{cD/3}$ , we define  $\mathcal{G}_i$  to be the subfamily of  $\tilde{\mathcal{G}}$  consisting of members satisfying Condition 2.12. We define  $\mathcal{G}'_i$  to be the family of birational counterparts of members of  $\mathcal{G}_i$ .

Note that  $\mathcal{G}_i$  and  $\mathcal{G}'_i$  are non-empty open subset of  $\tilde{\mathcal{G}}_i$  and  $\tilde{\mathcal{G}}'_i$ , respectively, and  $\mathcal{G}_i = \tilde{\mathcal{G}}_i$ ,  $\mathcal{G}'_i = \tilde{\mathcal{G}}'_i$  for  $i \notin \{6, 7, 9, 10, 16, 18, 21\}$ . Note also that Condition 2.12 is only used to exclude curves of low degree in Section 10.

**2.5. Structure of the proof.** We recall definitions of maximal extraction and maximal centers.

**Definition 2.14.** Let  $X$  be a  $\mathbb{Q}$ -Fano variety with Picard number 1 and let  $\varphi: Y \rightarrow X$  be a divisorial contraction with exceptional divisor  $E$ . We say that  $\varphi$  is a *maximal extraction* if there is a movable linear system  $\mathcal{H} \sim_{\mathbb{Q}} -nK_X$  such that

$$\frac{1}{n} > c(X, \mathcal{H}) = \frac{a_E(K_X)}{m_E(\mathcal{H})},$$

where  $m_E(\mathcal{H})$  is the multiplicity of  $\mathcal{H}$  along  $E$ ,  $a_E(K_X)$  is the discrepancy of  $K_X$  along  $E$ , and  $c(X, \mathcal{H})$  is the canonical threshold of the pair  $(X, \mathcal{H})$ . A closed subvariety  $\Gamma \subset X$  is called a *maximal center* if there is a maximal extraction centered along  $\Gamma$ .

In the rest of this paper, we prove the following.

**Theorem 2.15.** *Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ . Then, no nonsingular point and no curve on  $X'$  is a maximal center. Moreover, for each divisorial extraction  $\varphi: Y' \rightarrow X'$  centered at a singular point, one of the following holds.*

- (1)  $\varphi$  is not a maximal extraction.



- (2) *There is a birational involution  $\iota: X' \dashrightarrow X'$  that is a Sarkisov link starting with  $\varphi$ .*
- (3) *There is a Sarkisov link  $\sigma: X' \dashrightarrow X$  starting with  $\varphi$ .*

We note that Theorem 2.15 follows from Theorems 4.1, 4.4, 5.1, 5.6, 6.2, 6.7, 7.4, 8.7, 9.2 and 10.25. By [13, Lemma 2.32], Theorem 1.2 follows from Theorem 2.15 and [12, Theorem 1.3]. We explain the outline of this paper. In Section 3, we explain divisorial extractions centered at  $cA/n$  and  $cD/3$  points. In Sections 4, 5 and 6, we construct various Sarkisov links that are links from  $X'$  to  $X$  and birational involutions of  $X'$  centered at singular points. The construction of birational involutions centered at terminal quotient singular points is the same as that of [5] with a single exception. We need a hard construction due to [3] for the exceptional case. The construction of birational involutions centered at  $cA/n$  points is based on explicit global descriptions of divisorial extractions centered at  $cA/n$  points. This is quite similar to [1]. The rest of the paper is devoted to exclusion. Nonsingular points, some terminal quotient singular points, some  $cA/n$  and  $cD/3$  points, and curves are excluded in Sections 7, 8, 9 and 10, respectively. We exclude points and most of the curves by the combination of methods in [5, 3]. We refer the readers to [13, Section 2.4] for various excluding methods. A large volume of the paper is devoted to excluding curves of low degree passing through the  $cA/n$  point. The method is simple but we need careful and quite complicated computations. Finally, the table in Section 11, *the big table*, summarizes the results from which one can see what happens at each singular point.

### 3. DIVISORIAL EXTRACTIONS CENTERED AT $cA/n$ AND $cD/3$ POINTS

In this section, we describe divisorial extractions centered at the point  $\mathbf{p} := \mathbf{p}_4$  of a member  $X'$  of the family  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ . This is based on the classification results due to Hayakawa [6] and Kawakita [8, 9].

**3.1.  $cA/n$  point.** Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^*$  and  $\mathbf{p} = \mathbf{p}_4$ . Throughout this section, we assume that  $X'$  is defined by a standard polynomial  $F' = w^2x_2x_3 + wf + g$  in  $\mathbb{P}(1, n, a_2, a_3, n)$ . Set  $d := \deg F'$ . Note that  $d \geq 5$ . Moreover  $a_2 + a_3 \equiv 0 \pmod{n}$  if  $n > 1$ .

We see that a general hyperplane section of the index 1 cover of the singularity  $(X', \mathbf{p})$  is a du Val singularity of type  $A_{d-n-1}$ . If  $n \neq 2, 4$ , then the classification of 3-dimensional terminal singularities immediately implies that  $(X', \mathbf{p})$  is of type  $cA/n$ . If  $n = 2$  or  $4$ , then  $(X', \mathbf{p})$  is of type  $cA/n$ ,  $cAx/2$  or  $cAx/4$ . It is straightforward to see that  $(X', \mathbf{p})$  is not of type  $cAx/2$  and  $cAx/4$ , and hence  $(X', \mathbf{p})$  is of type  $cA/n$ . Since  $(X', \mathbf{p}_4)$  is of type  $cA/n$ , we have an identification,

$$(\mathbf{p} \in X') \cong (o \in (s_1s_2 + h(s_3, s_4) = 0)/\mathbb{Z}_n(a_2, -a_2, 1, 0)),$$

for some  $h(s_3, s_4) \in \mathbb{C}\{s_3, s_4\}$ .

**Lemma 3.1.** *Let  $\mathbb{P} := \mathbb{P}(c_1, c_2, c_3, c_4)$  be a weighted projective space with homogeneous coordinates  $s_1, s_2, s_3$  and  $s_4$ . Suppose that  $c_1 \geq c_2$ ,  $c_1 > c_i$  for  $i = 3, 4$ , and there is an automorphism  $\sigma$  of  $\mathbb{P}$  that induces an isomorphism*

$$\sigma|_{H_1}: H_1 := (s_0s_1 + g_1(s_2, s_3) = 0) \xrightarrow{\cong} H_2 := (s_0s_1 + g_2(s_2, s_3) = 0)$$

*between weighted hypersurfaces in  $\mathbb{P}$ , where  $g_i$  is homogeneous of degree  $c_0 + c_1$ . Then there is an automorphism  $\tau$  of  $\mathbb{P}(c_3, c_4)$  such that  $g_1 = \tau^*g_2$ .*

*Proof.* Let  $\sigma^*$  be the automorphism of  $\mathbb{C}[s_1, s_2, s_3, s_4]$  induced by  $\sigma$ . We have

$$(1) \quad \sigma^*(s_1 s_2 + g_2) = \alpha(s_1 s_2 + g_1)$$

for some  $\alpha \neq 0$ . After re-scaling coordinates, we may assume  $\alpha = 1$ .

We first treat the case where  $c_0 > c_1$ . After re-scaling  $s_0$ , we have  $\sigma^* s_1 = s_1 + a$  for some  $a \in \mathbb{C}[s_2, s_3, s_4]$ , and  $\sigma^* s_i$  does not involve  $s_1$  for  $i = 2, 3, 4$  since  $c_1 > c_i$ . By comparing the terms involving  $s_1$  in (1), we have  $\sigma^* s_2 = s_2$  and thus  $g_1 = a s_2 + \sigma^* g_2$ . It follows that  $\sigma$  restricts to an automorphism  $\bar{\sigma}$  of  $(s_2 = 0) \cong \mathbb{P}(c_1, c_3, c_4)$ . Moreover, since  $\sigma^* s_i$  does not involve  $s_1$  for  $i = 3, 4$ , the correspondence  $s_i \mapsto \bar{\sigma}^* s_i$  for  $i = 3, 4$  defines an automorphism of  $\mathbb{P}(c_3, c_4)$ , which we denote by  $\tau$ . By the construction, we have  $g_1 = \tau^* g_2$ .

We treat the case where  $c_1 = c_2$ . We have  $\sigma^* s_1 = \alpha_1 s_1 + \alpha_2 s_2 + a$  and  $\sigma^* s_2 = \beta_1 s_1 + \beta_2 s_2 + b$  for some  $\alpha_i, \beta_i \in \mathbb{C}$  and  $a, b \in \mathbb{C}[s_3, s_4]$ . Note that  $\sigma^* s_i \in \mathbb{C}[s_3, s_4]$  for  $i = 3, 4$  since  $c_3, c_4 < c_1 = c_2$ . We have  $(\alpha_1, \beta_1) \neq (0, 0)$  since  $\sigma$  is an automorphism. Possibly interchanging  $s_1$  and  $s_2$ , we may assume  $\alpha_1 \neq 0$ . Then, by comparing terms involving  $s_1$  and  $s_2$  in (1), we have  $\alpha_2 = \beta_1 = 0$ ,  $\alpha_1 \beta_2 = 1$ ,  $a = b = 0$  and  $g_1 = \sigma^* g_2$ . Thus  $\sigma$  restricts to an automorphism  $\tau$  of  $(s_1 = s_2 = 0) \cong \mathbb{P}(c_3, c_4)$  and we have  $g_1 = \tau^* g_2$ . This completes the proof.  $\square$

**Lemma 3.2.** *We have an equivalence*

$$(\mathfrak{p} \in X') \cong (o \in (s_1 s_2 + h(s_3, s_4) = 0) / \mathbb{Z}_n(a_2, -a_2, 1, 0))$$

*of singularities, where the lowest weight part of  $h$  with respect to the weight  $\text{wt}(s_3, s_4) = (1, n)$  is  $h_{d-n} = f(s_3, s_4, 0, 0)$ .*

*Proof.* Let  $\varphi: Y' \rightarrow X'$  be the weighted blowup with  $\text{wt}(x_0, x_1, x_2, x_3) = \frac{1}{n}(1, n, a_2 + n, a_3)$  at  $\mathfrak{p}$  with exceptional divisor  $E$ . It is proved in [12, Section 4.2] that  $\varphi$  is a divisorial contraction (see also Section 4). We have an isomorphism

$$E \cong (x_2 x_3 + f(x_0, x_1, 0, x_3) = 0) \subset \mathbb{P}(1, n, a_2 + n, a_3).$$

By filtering off terms divisible by  $x_3$  and then replacing  $x_2$ , we see that  $E \cong (x_2 x_3 + f(x_0, x_1, 0, 0) = 0)$ . Let

$$(\mathfrak{p} \in X') \cong (o \in (s_1 s_2 + h(s_3, s_4) = 0) / \mathbb{Z}_n(a_2, -a_2, 1, 0)).$$

be any identification of the  $cA/n$  point  $\mathfrak{p}$ , where  $h(s_3, s_4) \in \mathbb{C}\{s_3, s_4\}$ . Let  $\psi$  be the divisorial extraction of  $(s_1 s_2 + h = 0) / \mathbb{Z}_n$  which corresponds to  $\varphi$ , and let  $F$  be the  $\psi$ -exceptional divisor.  $\psi$  is a weighted blowup with  $\text{wt}(s_1, s_2, s_3, s_4) = \frac{1}{n}(c_1, c_2, c_3, c_4)$  for some  $c_1, \dots, c_4$ . The identification of singularities induces an isomorphism

$$\sigma: \mathbb{P}(1, n, a_2 + n, a_3) \rightarrow \mathbb{P}(c_1, c_2, c_3, c_4)$$

which restricts to an isomorphism  $\sigma|_E: E \rightarrow F$  between exceptional divisors. In particular, we have  $\{c_1, c_2, c_3, c_4\} = \{1, n, a_2 + n, a_3\}$  and we may assume that  $c_1 = a_2 + n$ ,  $c_2 = a_3$ ,  $c_3 = 1$  and  $c_4 = n$  after interchanging  $s_1$  and  $s_2$ , and  $s_3$  and  $s_4$ . Here  $F = (s_1 s_2 + h_m(s_3, s_4) = 0)$ , where  $h_m$  is the lowest weight part of  $h$ . Since  $a_2 + n > 1, n$ , we can apply Lemma 3.1 for the isomorphism  $\sigma$  and there is an isomorphism  $\tau: \mathbb{P}(1, n) \rightarrow \mathbb{P}(c_3, c_4)$  such that  $\tau^* h_m = f(x_0, x_1, 0, 0)$ . We see that  $\tau$  extends to an  $\mathbb{Z}_n$ -equivariant automorphism of  $\mathbb{A}^4$  with coordinates  $s_1, s_2, s_3, s_4$  by setting  $\tau^* s_i = s_i$  for  $i = 1, 2$ . Thus, by replacing the germ  $(s_1 s_2 + h = 0) / \mathbb{Z}_n$  with the automorphic image  $(s_1 s_2 + \tau^* h = 0) / \mathbb{Z}_n$ , we see that  $m = \deg f = d - n$  and  $h_{d-n} = f(s_3, s_4, 0, 0)$ .  $\square$

**Definition 3.3.** Under the above identification, for positive integers  $r_1$  and  $r_2$ , let

$$\varphi_{(r_1, r_2)}: Y'_{(r_1, r_2)} \rightarrow X'$$

be the birational morphism that is the weighted blowup of  $X'$  at  $\mathfrak{p}$  with weight  $\text{wt}(s_1, s_2, s_3, s_4) = \frac{1}{n}(r_1, r_2, 1, n)$ . We call  $\varphi_{(r_1, r_2)}$  the  $\frac{1}{n}(r_1, r_2)$ -blowup.

**Lemma 3.4.** (1) Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/1}^*$ . Then,

$$\{\varphi_{(r_1, r_2)} \mid r_1, r_2 > 0, r_1 + r_2 = d - 1\}$$

are the divisorial extractions centered at  $\mathfrak{p}$ .

(2) Let  $X' \in \mathcal{G}'_i$  be a member of  $i \in I_{cA/n}$  with  $n \geq 2$ . Then,

$$\{\varphi_{(r_1, r_2)} \mid r_1, r_2 > 0, r_1 + r_2 = d - n, r_1 \equiv a_2 \pmod{n}\},$$

are the divisorial extractions centered at  $\mathfrak{p}$ .

*Proof.* (2) follows immediately from [6, §6]. We will prove (1). According to the classification [8, Theorem 1.13], we need to show that a weighted blowup “of type  $(r_1, r_2, k, 1)$ ” cannot be a divisorial extraction of  $(\mathfrak{p} \in X')$  for  $k \geq 2$ . Suppose that  $(\mathfrak{p} \in X')$  admits such an extraction. This means that there are  $k \geq 2$  and an identification

$$(\mathfrak{p} \in X') \cong (o \in (s_1 s_2 + h'(s_3, s_4) = 0))$$

for some  $h' \in \mathbb{C}\{s_3, s_4\}$  such that  $s_3^{d-n} \in h'$  and  $s_3^i s_4^j \notin h'$  for  $2ki + j < k(d - n)$ . In this case, the weighted blowup with  $\text{wt}(s_1, s_2, s_3, s_4) = (r_1, r_2, k, 1)$  is a divisorial extraction of  $(\mathfrak{p} \in X')$  for any  $r_1, r_2$  such that  $r_1 + r_2$  is divisible by  $k$  and  $k$  is co-prime to  $r_1$  and  $r_2$ . But, in this case,  $Y'_{(r_1, r_2)}$  has a (unique)  $cA$  point along the exceptional divisor for any  $r_1, r_2$  such that  $r_1 + r_2 = d - n$ . On the other hand, by [12, Section 4.2], there is a pair  $(r_1, r_2)$  with  $r_1 + r_2 = d - n$  such that  $Y'_{(r_1, r_2)}$  has only terminal quotient singularities. This is a contradiction and the proof is completed.  $\square$

**3.2.  $cD/3$  point.** Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cD/3} = \{61, 62\}$  and  $\mathfrak{p} = \mathfrak{p}_4 \in X'$  the  $cD/3$  point. The singularity  $(X', \mathfrak{p})$  is indeed of type  $cD/3$  since a general hyperplane section of index 1 cover of  $(X', \mathfrak{p})$  is of type  $D$ .

**Lemma 3.5.** Suppose that  $i = 61$  (resp. 62) and  $X'$  is defined by a standard polynomial. Then the weighted blowup  $\varphi': Y' \rightarrow X'$  of  $X'$  at  $\mathfrak{p}$ , which is defined as the weighted blowup of  $X'$  at  $\mathfrak{p}$  with  $\text{wt}(x_0, x_1, y, z) = \frac{1}{3}(4, 1, 5, 6)$  (resp.  $\text{wt}(x, y, z, t) = \frac{1}{3}(1, 6, 4, 5)$ ), is the unique divisorial extraction centered at  $\mathfrak{p}$ .

*Proof.* It is proved in [12] that  $\varphi: Y' \rightarrow X'$  is an extremal extraction. According to the classification of extremal divisorial extraction of  $cD/3$  point in [6, §9], every extremal divisorial extraction centered at  $\mathfrak{p}$  is a weighted blowup and if it admits a weighted blowup with weight  $(1, 4, 5, 6)$  as an extremal divisorial extraction, then it is the unique extremal divisorial extraction of  $\mathfrak{p}$ .  $\square$

#### 4. SARKISOV LINKS BETWEEN $X$ AND $X'$

Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$  and  $\mathfrak{p} = \mathfrak{p}_4$ . We will show that there is a Sarkisov link  $X' \dashrightarrow X$  to the birational counterpart  $X \in \mathcal{G}_i$  starting with each  $\varphi_{(k, l)}$ -blowup marked  $X' \dashrightarrow X \ni \frac{1}{r}(\alpha, \beta, \gamma)$ . The mark  $\frac{1}{r}(\alpha, \beta, \gamma)$  indicates that the link  $X' \dashrightarrow X$  ends with the Kawamata blowup of  $X$  at a singular point of type  $\frac{1}{r}(\alpha, \beta, \gamma)$ .

**Theorem 4.1.** *Suppose  $i \in I_{cD/3} = \{61, 62\}$ . Then, there is a Sarkisov link  $X' \dashrightarrow X$  starting with the unique divisorial extraction centered at  $\mathfrak{p}$ .*

*Proof.* In [12, Theorem 4.10], we constructed a Sarkisov link  $X \dashrightarrow X'$  which ends with the divisorial contraction centered at  $\mathfrak{p}$ . Thus its inverse  $X' \dashrightarrow X$  is the desired one.  $\square$

In the following, we assume that  $i \in I_{cA/n}^*$  and  $X' \subset \mathbb{P}(1, n, a_2, a_3, n)$  is defined by a standard polynomial  $F' = w^2 x_2 x_3 + wf + g$ . Let  $X \in \mathcal{G}_i$  be the birational counterpart. In [12, Section 4.2], Sarkisov links  $X \dashrightarrow X'$  are constructed. Each link  $X \dashrightarrow X'$  ends with a divisorial extraction centered at  $\mathfrak{p}$ . Such an extraction is a weighted blowup and the complete description is given. We recall the construction.

Let  $\varphi': Y' \rightarrow X'$  be the weighted blowup with  $\text{wt}(x_0, x_1, x_2, x_3) = \frac{1}{n}(1, n, a_2 + n, a_3)$  (resp.  $\frac{1}{n}(1, n, a_2, a_3 + n)$ ). It is proved in [12, Section 4.2] that  $\varphi$  is a divisorial extraction and hence we have  $\varphi' = \varphi_{(a_2+n, a_3)}$  (resp.  $\varphi_{(a_2, a_3+n)}$ ). We define

$$Z := (x_4(x_4 x_3 + f) + x_2 g = 0) \subset \mathbb{P}(1, n, a_2, a_3, a_2 + n),$$

$$(\text{resp. } Z := (x_5(x_5 x_2 + f) + x_3 g = 0) \subset \mathbb{P}(1, n, a_2, a_3, a_3 + n)),$$

where  $\deg x_4 = a_2 + n$  and  $\deg x_5 = a_3 + n$ . By taking the ratio in two ways

$$x_5 := -\frac{x_4 x_3 + f}{x_2} = \frac{g}{x_4} \text{ and } w := \frac{x_4}{x_2} = -\frac{g}{x_3 + f},$$

$$\left( \text{resp. } x_4 := -\frac{x_5 x_2 + f}{x_3} = \frac{g}{x_5} \text{ and } w := \frac{x_5}{x_3} = -\frac{g}{x_5 x_2 + f} \right),$$

we obtain birational maps  $X \dashrightarrow Z$  and  $X' \dashrightarrow Z$ . The Kawamata blowup  $\varphi: Y \rightarrow X$  at  $\mathfrak{p}_5$  (resp.  $\mathfrak{p}_4$ ) and  $\varphi$  resolves the indeterminacies of  $X \dashrightarrow Z$  and  $X' \dashrightarrow Z$ , respectively, and we have the following diagram

$$\begin{array}{ccccc} Y' & \dashrightarrow & \tau & \dashrightarrow & Y \\ \varphi' \downarrow & & & & \downarrow \varphi \\ & & Z & & X \end{array}$$

where  $\tau$  is a flop. This is the Sarkisov link, denoted by  $\sigma_{(a_2+n, a_3)}$  (resp.  $\sigma_{(a_2, a_3+n)}$ ), starting with  $\varphi_{(a_2+n, a_3)}$  (resp.  $\varphi_{(a_2, a_3+n)}$ ).

If  $i = \{7, 10\}$ , then the above construction gives all the Sarkisov links  $X' \dashrightarrow X$  centered at  $\mathfrak{p}$ .

Suppose  $i \neq \{7, 10\}$ , that is,

$$i \in \{6, 9, 16, 18, 22, 26, 28, 33, 44, 48, 57\}.$$

In the following, let  $(k, l)$  be either  $(a_2 + n, a_3)$  or  $(a_2, a_3 + n)$ . We have constructed a Sarkisov link  $X' \dashrightarrow X$  starting with  $\frac{1}{n}(k, l)$ -blowup. We see that  $X'$  admits  $\frac{1}{n}(l, k)$ -blowup. We will explain a relation between  $\varphi_{(k, l)}$  and  $\varphi_{(l, k)}$  and show that there is a Sarkisov link  $X' \dashrightarrow X$  starting with  $\varphi_{(l, k)}$ .

Suppose  $i \in \{9, 22, 28, 33, 48, 57\}$ . Then  $\deg F' = 2a_3$  and we can write

$$F' = w^2 x_2 x_3 + w(x_3 a + b) + x_3^2 + x_3 c + d$$

for some  $a, b, c, d \in \mathbb{C}[x_0, x_1, x_2]$ .

**Definition 4.2.** For  $i \in \{9, 22, 28, 33, 48, 57\}$ , we define  $\mu$  to be the biregular involution of  $X'$  defined by the replacement  $x_3 \mapsto -x_3 - w^2 x_2 - wa - c$ .

We see that  $\mu(\mathfrak{p}) = \mathfrak{p}$  and that the composite  $\mu \circ \varphi_{(k,l)}$  defines the  $\varphi_{(l,k)}$ -blowup. The diagram

$$\begin{array}{ccccc}
 & Y' & \xrightarrow{\text{flop}} & Y & \\
 \varphi_{(l,k)} \swarrow & \downarrow \varphi_{(k,l)} & & \downarrow \varphi & \\
 X' & \xrightarrow{\mu} & X' & \xrightarrow{\sigma_{(k,l)}} & X \\
 & \searrow \sigma_{(l,k)} & & & 
 \end{array}$$

gives the Sarkisov link  $\sigma_{(l,k)}: X' \dashrightarrow X$  starting with  $\varphi_{(l,k)}$ .

Suppose  $i \in \{6, 16, 18, 26, 44\}$ . Then,  $\deg F' = 2a_3 + n$  and we can write

$$F' = w^2 x_2 x_3 + w(x_3^2 + x_3 a + b) + x_3 c + d$$

for some  $a, b, c, d \in \mathbb{C}[x_0, x_1, x_2]$ . Here, a priori, there is a term  $x_3^2 e$  in  $F'$  for some  $e \in \mathbb{C}[x_0, x_1, x_2]$  but we can eliminate the term by replacing  $w$  with  $w - e$ . The birational counterpart  $X \in \mathcal{G}_i$  is defined by

$$F_1 = x_5 x_2 + x_4 x_3 + (x_3^2 + x_3 a + b)$$

$$F_2 = x_5 x_4 - (x_3 c + d)$$

in  $\mathbb{P}(1, n, a_2, a_3, a_2 + n, a_3 + n)$ .

**Definition 4.3.** Suppose  $i \in \{6, 16, 18, 26, 44\}$ . We define

$$\begin{aligned}
 \tilde{F}' &:= F'(x_0, x_1, x_2, -x_3 - wx_2 - a, w) \\
 &= w^2 x_2 x_3 + w(x_3^2 + x_3 a + b - x_2 c) - x_3 c + d - ac
 \end{aligned}$$

and then define  $\tilde{X}'$  to be the weighted hypersurface defined by  $\tilde{F}'$ . We denote by  $\nu': X' \rightarrow \tilde{X}'$  the isomorphism defined by the replacement  $x_3 \mapsto -x_3 - wx_2 - a$ .

We define

$$\begin{aligned}
 \tilde{F}_1 &:= F_1(x_0, x_1, x_2, -x_3 - x_4 - a, x_4, x_5 + c) \\
 &= x_5 x_2 + x_4 x_3 + (x_3^2 + x_3 a + b - x_2 c) \\
 \tilde{F}_2 &:= F_2(x_0, x_1, x_2, -x_3 - x_4 - a, x_4, x_5 + c) \\
 &= x_5 x_4 - (-x_3 c + d - ac)
 \end{aligned}$$

and then define  $\tilde{X}$  to be the WCI defined by  $\tilde{F}_1 = \tilde{F}_2 = 0$ . We denote by  $\mu: X \rightarrow \tilde{X}$  the isomorphism defined by the replacements  $x_3 \mapsto -x_3 - x_4 - a$  and  $x_5 \mapsto x_5 + c$ .

Since  $\tilde{X}' \in \mathcal{G}'_i$  and  $\tilde{X}$  is the birational counterpart of  $\tilde{X}'$ , there is a Sarkisov link  $\tilde{\sigma}_{(k,l)}: \tilde{X}' \dashrightarrow \tilde{X}$  starting with  $(k, l)$ -blowup  $\tilde{\varphi}_{(k,l)}$  of  $\tilde{X}'$  and ending with Kawamata blowup  $\tilde{\varphi}: \tilde{Y} \rightarrow \tilde{X}$ . We see that  $\varphi_{(l,k)} = \nu'^{-1} \circ \tilde{\varphi}_{(k,l)}$  and that the composite  $\varphi := \nu^{-1} \circ \tilde{\varphi}$  is the Kawamata blowup of  $X$ . Therefore the diagram

$$\begin{array}{ccccc}
 & \tilde{Y}' & \xrightarrow{\text{flop}} & \tilde{Y} & \\
 \varphi_{(l,k)} \swarrow & \downarrow \tilde{\varphi}_{(k,l)} & & \downarrow \tilde{\varphi} & \searrow \varphi_{\mathfrak{p}} \\
 X' & \xrightarrow{\nu'} & \tilde{X}' & \xrightarrow{\tilde{\sigma}_{(k,l)}} & \tilde{X} \xleftarrow{\nu} X \\
 & \searrow \sigma_{(l,k)} & & & 
 \end{array}$$

gives the Sarkisov link  $\sigma_{(l,k)}: X' \dashrightarrow X$  starting with  $\varphi_{(l,k)}$  and ending with  $\varphi$ . As a conclusion, we have the following.

**Theorem 4.4.** *Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^*$  and let  $\varphi_{(k,l)}$  be a divisorial extraction centered at  $\mathbf{p} = \mathbf{p}_4$  marked  $X' \dashrightarrow X$  in the big table. Then there is a Sarkisov link  $\sigma_{(k,l)}: X' \dashrightarrow X$  starting with  $\varphi_{(k,l)}$ .*

## 5. BIRATIONAL INVOLUTIONS CENTERED AT QUOTIENT SINGULAR POINTS

In this section, we construct a birational involution of  $X' \in \mathcal{G}'_i$  that is a Sarkisov link centered at a suitable terminal quotient singular point. Throughout this section, let  $\mathbf{p} \in X'$  be a terminal quotient singular point with non-empty third column in the big table and let  $\varphi': Y' \rightarrow X'$  be the Kawamata blowup of  $X'$  at  $\mathbf{p}$  with exceptional divisor  $E$ . For a divisor or a curve  $\Delta$  on  $X'$ , we denote by  $\tilde{\Delta}$  the proper transform of  $\Delta$  on  $Y'$ .

### 5.1. Quadratic involutions.

**Theorem 5.1.** *Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$  and  $\mathbf{p} \in X'$  a terminal quotient singular point marked *Q.I.* in the third column. Then there exists a birational involution of  $X'$  that is a Sarkisov link centered at  $\mathbf{p}$ .*

*Proof.* Let  $\mathbb{P}(a_0, \dots, a_4)$  be the ambient space of  $X'$  with coordinates  $x_0, \dots, x_3, w$ . After replacing coordinates, we may assume  $\mathbf{p} = \mathbf{p}_j$  for some  $j \neq 4$ . Then, after replacing coordinates, the defining polynomial can be written as  $F' = x_j^2 x_k + x_j h_1 + h_2$ , where  $k \neq j$  and  $h_1, h_2$  are polynomials in  $x_0, \dots, x_3, w$  that do not involve  $x_j$ . It then follows from [5, Theorem 4.9] that there is a birational involution of  $X'$  that is a Sarkisov link centered at  $\mathbf{p}$ .  $\square$

**5.2. Family  $\mathcal{G}'_{18}$  and the point of type  $\frac{1}{2}(1, 1, 1)$ .** Let  $X' = X'_8 \subset \mathbb{P}(1, 1, 2, 3, 2)$  be a member of  $\mathcal{G}'_{18}$  and  $\mathbf{p} \in X'$  the singular point of type  $\frac{1}{2}(1, 1, 1)$ .

The defining polynomial of  $X'$  is of the form  $F' = w^2 x_0 z + w f_6 + g_8$ , where  $f_6, g_8 \in \mathbb{C}[x_0, x_1, y, z]$ . If  $y^3 \notin f_6$ , then there is no  $\frac{1}{2}(1, 1, 1)$  point on  $X'$ . Hence we may assume that  $y^3 \in f_6$ . After replacing  $w$  suitably, we assume that  $\mathbf{p} = \mathbf{p}_2$  and there is no monomial in  $g_8$  that is divisible by  $y^3$ .

We first treat the case where  $z^2 y \in g_8$ . By replacing coordinates suitably, we have

$$F' = yz^2 + a_5 z - wy^3 - b_4 y^2 - c_6 y + d_8,$$

where  $a_5, b_4, c_6, d_8 \in \mathbb{C}[x_0, x_1, w]$ . By [5, Theorem 4.13], the sections

$$u := z^2 - wy^2 - b_4 y - c_6 \text{ and } v := uz + a_5 wy + a_5 b_4$$

lift to plurianticanonical sections of  $Y'$ . Moreover, the anticanonical model  $Z'$  of  $Y'$  is the weighted hypersurface defined by the equation

$$-v^2 + a_6 b_4 v + u^3 + u^2 c_6 - (b_4 d_8 + a_5^2 w)v + (-a_5^2 c_6 + d_8^2)w = 0$$

in  $\mathbb{P}(1_{x_0}, 1_{x_1}, 2_w, 6_u, 9_v)$  and the corresponding map  $\psi': Y' \dashrightarrow Z'$  is a morphism. By [13, Lemma 3.2], either  $\mathbf{p}$  is not a maximal center or there is a birational involution of  $X'$  that is a Sarkisov link centered at  $\mathbf{p}$ .

In the following, we treat the case where  $z^2 y \notin g_8$ . We will construct a birational involution of a suitable model of  $Y'$  and observe that the induced birational involution of  $X'$  gives a Sarkisov link starting with  $\varphi$ . This kind of involution is called an *invisible involution* in [3] and its construction is first introduced there. One can find its construction in a relatively general setting in [13, Section 7.1] and the rest of this section is to verify [13, Condition 7.1]. This involves complicated computations

and a reader not interested in special members may skip this part since this does not happen for a general  $X' \in \mathcal{G}'_{33}$ .

After re-scaling  $y, z, w$ , we assume that the coefficients of  $y^3$  and  $z^2$  in  $f_6$  are both 1. In this case, either  $y^2zx_0 \in g_8$  or  $y^2zx_1 \in g_8$  because otherwise  $X'$  is not quasismooth at  $(0:0:-1:1:0)$ . We see that  $x_0, x_1, z$  vanish along  $E$  to order  $1/2$  and  $w$  vanishes along  $E$  to order  $2/2$ . For  $\lambda, \mu \in \mathbb{C}$ , we define  $S_\lambda := (x_1 - \lambda x_0 = 0)_{X'}$  and  $T_\mu := (w - \mu x_0^2 = 0)_{X'}$ . We see that  $S_\lambda$  is normal for a general  $\lambda$ . We set

$$\bar{F}'_{\lambda,\mu} := F(x_0, \lambda x_0, y, z, \mu x_0^2) \in \mathbb{C}[x_0, y, z].$$

We see that  $\bar{F}'_{\lambda,\mu}$  is divisible by  $x_0$  but not by  $x_0^2$  since  $y^2zx_0 \in g_8$  or  $y^2zx_1 \in g_8$ . We have  $T_\mu|_{S_\lambda} = \Gamma + C_{\lambda,\mu}$ , where  $\Gamma = (x_0 = x_1 = w = 0)$  and

$$C_{\lambda,\mu} = (x_1 - \lambda x_0 = w - \mu x_0^2 = \bar{F}'_{\lambda,\mu}/x_0 = 0).$$

Let  $Z' \rightarrow Y'$  be the Kawamata blowup of  $Y'$  at the  $\frac{1}{3}(1, 1, 2)$  point that is the inverse image of  $\mathbf{p}_3$  by  $\varphi$  with exceptional divisor  $F \cong \mathbb{P}(1, 1, 2)$ . Let  $W \rightarrow Z'$  be the Kawamata blowup of  $Z'$  at the  $\frac{1}{2}(1, 1, 1)$  point lying on  $F$  with exceptional divisor  $G$ . For a curve or a divisor  $\Delta$  on  $X', Y'$  or  $Z'$ , we denote by  $\hat{\Delta}$  the proper transform of  $\Delta$  on  $W$ .

**Lemma 5.2.** *We have*

$$(-K_{Y'} \cdot \tilde{C}_{\lambda,\mu}) = \frac{2}{3}, (-K_W \cdot \hat{C}_{\lambda,\mu}) = 0, (-K_W \cdot \hat{\Gamma}) = -1$$

and

$$(\hat{E} \cdot \hat{C}_{\lambda,\mu}) = 1, (\hat{F} \cdot \hat{C}_{\lambda,\mu}) = 0, (G \cdot \hat{C}_{\lambda,\mu}) = 1.$$

*Proof.* In this proof, we write  $S = S_\lambda$ ,  $T = T_\mu$  and  $C = C_{\lambda,\mu}$  for simplicity. We see that  $\tilde{\Gamma}$  intersects  $E$  at one point so that  $(E \cdot \tilde{\Gamma}) = 1$ . For a curve or a divisor  $\Delta$  on  $X'$  or  $Y'$ , we denote by  $\check{\Delta}$  the proper transform of  $\Delta$  on  $Z'$ . We see that  $\check{\Gamma}$  intersects  $F$  at the  $\frac{1}{2}(1, 1, 1)$  point. We compute the intersection number  $(F \cdot \check{\Gamma})$  by considering a suitable weighted blowup of the ambient space of  $Y'$ . We may choose  $x_0, x_1, y, z$  as local orbifold coordinates of  $Y'$  at the  $\frac{1}{3}(1, 1, 2)$  point. The weighted blowup of the ambient space with weight  $\text{wt}(x_0, x_1, y, w) = \frac{1}{3}(1, 1, 2, 2)$  restricts to the Kawamata blowup  $Z' \rightarrow Y'$ . Since  $\tilde{\Gamma}$  is defined by  $(x_0 = x_1 = w = 0)$ , we have

$$(F \cdot \check{\Gamma}) = (F \cdot -\frac{1}{3}F \cdot -\frac{1}{3}F \cdot -\frac{2}{3}F) = -\frac{2}{3^3} \times \frac{(-3)^3}{2 \times 2} = \frac{1}{2}.$$

In the above equation, we think of  $F$  as the exceptional divisor of the weighted blowup of the ambient space, which is isomorphic to  $\mathbb{P}(1, 1, 2, 2)$ . We see that  $\hat{\Gamma}$  intersects  $G$  at one point so that  $(G \cdot \hat{\Gamma}) = 1$ .

We have  $(-K_{Y'} \cdot \tilde{\Gamma}) = (-K_{X'} \cdot \Gamma) - \frac{1}{2}(E \cdot \tilde{\Gamma}) = -\frac{1}{3}$ . Similarly, we have  $(-K_Z \cdot \check{\Gamma}) = (-K_{Y'} \cdot \tilde{\Gamma}) - \frac{1}{3}(F \cdot \check{\Gamma}) = -\frac{1}{2}$  and  $(-K_Z \cdot \hat{\Gamma}) = (-K_Z \cdot \check{\Gamma}) - \frac{1}{2}(G \cdot \hat{\Gamma}) = -1$ . Since  $\tilde{S} \sim_{\mathbb{Q}} -K_{Y'}$ ,  $\tilde{T} \sim_{\mathbb{Q}} -2K_{Y'}$ ,  $\tilde{T}|_{\tilde{S}} = \tilde{\Gamma} + \tilde{C}$  and  $K_{Y'} = K_{X'} + \frac{1}{2}E$ , we have

$$2(-K_{Y'})^3 = (-K_{Y'} \cdot \tilde{T} \cdot \tilde{S}) = (-K_{Y'} \cdot \tilde{\Gamma}) + (-K_{Y'} \cdot \tilde{C}),$$

$$(-K_{Y'} \cdot \tilde{C}) = (-K_{X'} \cdot C) - \frac{1}{2}(E \cdot \tilde{C}).$$

It follows that  $(-K_{Y'} \cdot \tilde{C}) = 2/3$  and  $(E \cdot \tilde{C}) = 1$  since  $(E^3) = 4$  and  $(-K_{Y'})^3 = (-K_{X'})^3 - (1/2^3)(E^3) = 1/6$ . Note that  $(\hat{E} \cdot \hat{C}) = (E \cdot \tilde{C}) = 1$  since  $W \rightarrow Y'$



is an isomorphism over an open set which entirely contains  $E$ . Since  $\check{S} \sim_{\mathbb{Q}} -K_Z$ ,  $\check{T} \sim_{\mathbb{Q}} -2K_Z$ ,  $\check{T}|_{\check{S}} = \check{\Gamma} + \check{C}$  and  $K_Z = K_{Y'} + \frac{1}{3}F$ , we have

$$\begin{aligned} 2(-K_Z)^3 &= (-K_Z \cdot \check{T} \cdot \check{S}) = (-K_Z \cdot \check{\Gamma}) + (-K_Z \cdot \check{C}), \\ (-K_Z \cdot \check{C}) &= (-K_{Y'} \cdot \check{C}) - \frac{1}{3}(F \cdot \check{C}). \end{aligned}$$

It follows that  $(-K_Z \cdot \check{C}) = (F \cdot \check{C}) = 1/2$  since  $(F^3) = 9/2$  and  $(-K_Z)^3 = (-K_{Y'})^3 - (1/3^3)(F^3) = 0$ . Similarly, since  $\hat{S} \sim_{\mathbb{Q}} -K_W$ ,  $\hat{T} \sim_{\mathbb{Q}} -2K_W$ ,  $\hat{T}|_{\hat{S}} = \hat{\Gamma} + \hat{C}$  and  $K_W = K_Z + \frac{1}{2}G$ , we have

$$\begin{aligned} 2(-K_Z)^3 &= (-K_Z \cdot \hat{T} \cdot \hat{S}) = (-K_Z \cdot \hat{\Gamma}) + (-K_W \cdot \hat{C}), \\ (-K_W \cdot \hat{C}) &= (-K_Z \cdot \hat{C}) - \frac{1}{2}(G \cdot \hat{C}). \end{aligned}$$

It follows that  $(-K_W \cdot \hat{C}) = 0$  and  $(G \cdot \hat{C}) = 1$  since  $(G^3) = 4$  and  $(-K_W)^3 = (-K_Z)^3 - (1/2^3)(G^3) = -1/2$ . Finally, the pullback of  $F$  on  $W$  is the divisor  $\hat{F} + \frac{1}{2}G$ , hence we have  $(F \cdot \hat{C}) = (\hat{F} \cdot \hat{C}) + \frac{1}{2}(G \cdot \hat{C})$ , which implies  $(\hat{F} \cdot \hat{C}) = 0$ .  $\square$

**Lemma 5.3.** *If, for a general  $\lambda \in \mathbb{C}$ , there is  $\mu \in \mathbb{C}$  (depending on  $\lambda$ ) such that  $C_{\lambda,\mu}$  is reducible, then  $\mathfrak{p}$  is not a maximal center.*

*Proof.* We can write

$$\bar{F}'_{\lambda,\mu}/x_0 = \alpha x_0 z^2 + (\beta y^2 + \gamma y x_0^2 + \delta x_0^4)z + \mu y^3 x_0 + \varepsilon y^2 x_0^3 + \eta y x_0^5 + \theta x_0^7,$$

for some  $\alpha, \beta, \dots, \theta \in \mathbb{C}$ . Note that  $\alpha, \dots, \theta$  depend on  $\lambda$  and  $\mu$ , and  $\beta$  depends only on  $\lambda$ . Let  $\lambda$  be a general complex number so that  $\beta \neq 0$  and take  $\mu \in \mathbb{C}$  such that  $C_{\lambda,\mu}$  is reducible. Since  $\bar{F}'_{\lambda,\mu}/x_0$  is reducible, we have

$$\bar{F}'_{\lambda,\mu}/x_0 = (z + e_3)(\alpha x_0 z + e_4)$$

for some  $e_3, e_4 \in \mathbb{C}[x_0, y]$ . We have  $y^2 \in e_4$  since  $\beta \neq 0$ . It follows that  $C_{\lambda,\mu} = \Delta_1 + \Delta_2$ , where  $\Delta_1 = (x_1 - \lambda x_0 = w - \mu x_0^2 = z + e_3 = 0)$  and  $\Delta_2 = (x_1 - \lambda x_0 = w - \mu x_0^2 = \alpha x_0 z + e_4 = 0)$ . Note that  $\Delta_1$  is irreducible and  $\Delta_2$  does not pass through  $\mathfrak{p}$  since  $y^2 \in e_4$ . Thus  $(-K_{Y'} \cdot \tilde{\Delta}_2) = (-K_{X'} \cdot \Delta_2) = 2/3$ . This implies

$$(-K_{Y'} \cdot \tilde{\Delta}_1) = (-K_{Y'} \cdot \tilde{C}_{\lambda,\mu}) - (-K_{Y'} \cdot \tilde{\Delta}_2) = 0.$$

Clearly we have  $(E \cdot \tilde{\Delta}_1) > 0$ . Therefore, there are infinitely many irreducible curves on  $Y'$  that intersect  $-K_{Y'}$  non-positively and  $E$  negatively. By [13, Lemma 2.18],  $\mathfrak{p}$  is not a maximal center.  $\square$

Let  $\pi: X' \dashrightarrow \mathbb{P}(1_{x_0}, 1_{x_1}, 2_w)$  be the projection which is defined outside  $\Gamma$ . Let  $\mathcal{H} \subset |-2K_{X'}|$  be the linear system on  $X'$  generated by  $x_0^2, x_0 x_1, x_1^2$  and  $w$ , and let  $\mathcal{H}_{Y'}$ ,  $\mathcal{H}_W$  be the proper transform of  $\mathcal{H}$  on  $Y'$ ,  $W$ , respectively. We see that  $\mathcal{H}_{Y'} = |-2K_{Y'}|$  and  $\mathcal{H}_W = |-2K_W|$ . Let  $\pi_\lambda: S_\lambda \dashrightarrow \mathbb{P}(1, 2) \cong \mathbb{P}^1$  be the restriction of  $\pi$  to  $S_\lambda$  and  $\hat{\pi}: \hat{S}_\lambda \dashrightarrow \mathbb{P}^1$  be the composite of  $(\varphi' \circ \psi)|_{\hat{S}_\lambda}: \hat{S}_\lambda \rightarrow S_\lambda$  and  $\pi_\lambda$ . We set  $\hat{E}_\lambda = \hat{E}|_{\hat{S}_\lambda}$ ,  $\hat{F}_\lambda = \hat{F}|_{\hat{S}_\lambda}$  and  $\hat{G}_\lambda = G|_{\hat{S}_\lambda}$ . Note that  $\mathfrak{p}$  and  $\mathfrak{p}_3$  are the indeterminacy points of  $\pi_\lambda$ .

**Lemma 5.4.** *The base locus of  $\mathcal{H}_W$  is the curve  $\Gamma$  and the pair  $(W, \frac{1}{2}\mathcal{H}_W)$  is canonical.*

*Proof.* It is straightforward to see that  $\text{Bs } \mathcal{H}_W = \Gamma$ . We see that  $W$  is nonsingular along  $\Gamma$ , a general member of  $\mathcal{H}_W$  vanishes along  $\Gamma$  with multiplicity 1 and the blowing-up of  $W$  along  $\Gamma$  resolves the base locus of  $\mathcal{H}_W$ . This shows that  $(X, \frac{1}{2}\mathcal{H}_W)$  is canonical (in fact, terminal).  $\square$

**Lemma 5.5.** *The intersection matrix of curves in  $(x_0 = 0)|_{S_\lambda}$  is non-degenerate.*

*Proof.* We have  $(x_0 = 0)_{S_\lambda} = \Gamma + \Delta$ , where  $\Delta = (x_0 = x_1 = y^3 + z^2 = 0)$ . We have  $(\Gamma \cdot \Delta) = 1$  since  $\Gamma$  intersects  $\Delta$  at one nonsingular point. Since  $(-K_{X'})|_{S_\lambda} \sim_{\mathbb{Q}} (x_0 = 0)|_{S_\lambda} = \Gamma + \Delta$ , we have  $1/6 = (-K_{X'} \cdot \Gamma) = (\Gamma^2) + (\Gamma \cdot \Delta)$  and  $1/2 = (-K_{X'} \cdot \Delta) = (\Gamma \cdot \Delta) + (\Delta^2)$ . It follows that the intersection matrix

$$\begin{pmatrix} (\Gamma^2) & (\Gamma \cdot \Delta) \\ (\Gamma \cdot \Delta) & (\Delta^2) \end{pmatrix} = \begin{pmatrix} -5/6 & 1 \\ 1 & -1/2 \end{pmatrix}$$

is non-degenerate.  $\square$

By [13, Lemma 7.2], there is a birational involution of  $X'$  centered at  $\mathfrak{p}$ . As a conclusion, we have the following.

**Theorem 5.6.** *Let  $X'$  be a member of  $\mathcal{G}'_{18}$  and  $\mathfrak{p}$  a singular point of type  $\frac{1}{2}(1, 1, 1)$ . Then either  $\mathfrak{p}$  is not a maximal center or there is a birational involution of  $X'$  which is a Sarkisov link centered at  $\mathfrak{p}$ .*

## 6. BIRATIONAL INVOLUTIONS CENTERED AT $cA/n$ POINTS

Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^*$  and  $\mathfrak{p} = \mathfrak{p}_4$  the  $cA/n$  point. We treat 6 families, that is,

$$i \in \{10, 26, 33, 38, 48, 63\},$$

and show that there is a birational involution of  $X'$  starting with a divisorial extraction  $\varphi$  marked B.I. in the big table.

We briefly explain the argument of this section. Let  $\varphi: Y' \rightarrow X'$  be an extraction marked B.I. We first give an explicit global construction of  $\varphi$ , which enables us to give an explicit construction of anticanonical model  $Z'$  of  $Y'$ . We observe that the anticanonical map  $Y' \dashrightarrow Z'$  is a birational morphism and  $Z'$  admits a double cover onto a suitable WPS. By [13, Lemma 3.2], we conclude that either  $\mathfrak{p}$  is not a maximal center or there is a birational involution starting with  $\varphi$ .

**6.1. Families  $\mathcal{G}_i$  for  $i \in \{10, 26, 38, 48, 63\}$ .** We first treat the case where  $i \in \{10, 26, 48\}$ . The standard polynomial of  $X' \subset \mathbb{P}(1, n, a_2, a_3, n)$  can be written as

$$F' = w^2 x_2 x_3 + w(x_2^2 a + x_2 b + c) + x_2^3 + x_2^2 d + x_2 e + h,$$

where  $a, b, \dots, e, h \in \mathbb{C}[x_0, x_1, x_3]$ . Note that we have  $a_3 + 2n = 2a_2$ . Note also that we have  $n = 1$  in this case but we use  $n$  for a unified exposition.

We construct a divisorial extraction that corresponds to  $(a_2 - n, a_3 + 2n)$ -blowup. Filtering off terms divisible by  $x_2$ , the defining equation of  $X'$  is written as

$$(2) \quad F' = x_2 u + w c + h = 0,$$

where

$$(3) \quad u := w^2 x_3 + w(x_2 a + b) + x_2^2 + x_2 d + e.$$

Note that  $\deg u = a_3 + 2n$ . Let  $\mathbb{P} := \mathbb{P}(1, n, a_2, a_3, n, a_3 + 2n)$  be the WPS with coordinates  $x_0, \dots, x_3, w$  and  $u$ . Then,  $X'$  is a weighted complete intersection, defined by

the equations (2) and (3). Let  $\Phi: W \rightarrow \mathbb{P}$  be the weighted blowup at  $(0:0:0:0:1:0)$  with

$$\text{wt}(x_0, x_1, x_2, x_3, u) = \frac{1}{n}(1, n, a_2 - n, a_3, a_3 + 2n)$$

and  $Y'$  the proper transform of  $X'$  by  $\Phi$ . We denote by  $\varphi: Y' \rightarrow X'$  the induced birational morphism and by  $E$  its exceptional divisor.

**Lemma 6.1.** *The weighted blowup  $\varphi: Y' \rightarrow X'$  is a divisorial extraction centered at  $\mathfrak{p}$ .*

*Proof.* We will show that  $Y'$  has only terminal quotient singularities, which complete that proof. We see that  $X'$  is defined by

$$x_2u + wc + h = -u + w^2x_3 + w(x_2a + b) + x_2^2 + x_2d + e = 0$$

in  $\mathbb{P}$  and we have an isomorphism

$$E \cong (x_2u + c = x_3 + x_2a + x_2^2 = 0) \subset \mathbb{P}(1_{x_0}, n_{x_1}, (a_2 - n)_{x_2}, a_{3x_3}, (a_3 + 2n)_u).$$

We see that  $\deg a = a_2 - n < a_3$  and hence  $a$  does not involve the variable  $x_3$ . We have

$$J_\varphi = \begin{pmatrix} \frac{\partial c}{\partial x_0} & \frac{\partial c}{\partial x_1} & u & \frac{\partial c}{\partial x_3} & x_2 & h \\ x_2 \frac{\partial a}{\partial x_0} & x_2 \frac{\partial a}{\partial x_1} & a + 2x_2 & 1 & 0 & b + x_2d \end{pmatrix}.$$

We see that  $J_\varphi$  is of rank 2 outside the set

$$\begin{aligned} \Sigma &:= \left( x_2 = \frac{\partial c}{\partial x_0} = \frac{\partial c}{\partial x_1} = u - a \frac{\partial c}{\partial x_3} = h - b \frac{\partial c}{\partial x_3} = 0 \right) \cap E \\ &= \left( x_2 = x_3 = c = \frac{\partial c}{\partial x_0} = h - b \frac{\partial c}{\partial x_3} = \frac{\partial c}{\partial x_1} = u - a \frac{\partial c}{\partial x_3} = 0 \right). \end{aligned}$$

We claim that the system of equations

$$x_3 = c = \frac{\partial c}{\partial x_0} = \frac{\partial c}{\partial x_1} = h - b \frac{\partial c}{\partial x_3} = 0$$

has no non-trivial solution. Assume to the contrary that the above equations have a common solution  $(x_0, x_1, x_3) = (\alpha_0, \alpha_1, 0) \neq (0, 0, 0)$ . Let  $X$  be the birational counterpart of  $X'$  which is defined by

$$F_1 = x_5x_2 + x_4x_3 + (x_2^2a + x_2b + c) = 0,$$

$$F_2 = x_5x_4 - (x_2^3 + x_2^2d + x_2e + h) = 0,$$

in  $\mathbb{P}(1, n, a_2, a_3, a_2 + n, a_3 + n)$  and set  $\mathfrak{q} := (\alpha_0 : \alpha_1 : 0 : 0 : -\bar{c}' : -\bar{b})$ , where  $\bar{c}' = (\partial c / \partial x_3)(\alpha_0, \alpha_1, 0)$  and  $\bar{b} = b(\alpha_0, \alpha_1, 0)$ . Then,  $\mathfrak{q} \in X$  and  $\partial F_1 / \partial x_i$  vanishes at  $\mathfrak{q}$  for  $i = 0, 1, \dots, 5$ . Thus  $X$  is not quasismooth at  $\mathfrak{q}$  and this is a contradiction. The above claim implies that  $\Sigma = \emptyset$ . By Lemma 2.1,  $Y'$  has only cyclic quotient singularities. Straightforward computations in each instance show that  $Y'$  has only terminal quotient singularities (see Table 4 for the singularities of  $Y'$  along  $E$ ), which implies that  $\varphi$  is a divisorial extraction.  $\square$

We see that  $x_0, x_1, x_3$  and  $u$  lift to plurianticanonical sections on  $Y'$ . We construct one more section  $v$  that lifts to a plurianticanonical section on  $Y'$  and then determine the anticanonical map of  $Y'$ . Multiplying  $u$  to (3) and then eliminating  $x_2u = -wc - h$  by (2), we have

$$(4) \quad wv - u^2 + ue - h(x_2 + d) = 0,$$

TABLE 4. Singularities of  $Y'$  along  $E$

No.	Singular points
10	$1 \times \frac{1}{3}(1, 1, 2)$
26	$1 \times \frac{1}{2}(1, 1, 1)$
38	$1 \times \frac{1}{3}(1, 1, 2)$ if $z^3 \notin f_9$ , $1 \times \frac{1}{8}(1, 3, 5)$
48	$1 \times \frac{1}{3}(1, 1, 2)$ , $1 \times \frac{1}{8}(1, 1, 7)$
63	$1 \times \frac{1}{3}(1, 1, 2)$ if $y^4 \in f_{12}$ , $1 \times \frac{1}{10}(1, 3, 7)$

where

$$(5) \quad v := w(ux_3 - ac) + ub - cx_2 - cd - ah.$$

We have  $\deg v = 4a_2 - n$ . The sections  $u^2$ ,  $ue$  and  $hd$  vanish along  $E$  to order  $4a_2/n$ , and the section  $hx_2$  vanishes along  $E$  to order  $(4a_2 - n)/n$ . Hence, by (4),  $v$  vanishes along  $E$  to order at least  $(4a_2 - n)/n = \deg v/n$ . This shows that  $v$  lifts to a plurianticanonical section. Let

$$\psi: X' \dashrightarrow \mathbb{P}(1_{x_0}, n_{x_1}, a_{3x_3}, (a_3 + 2n)_u, (4a_2 - n)_v)$$

the rational map defined by plurianticanonical sections  $x_0, x_1, x_3, u, v$ . We see that the intersection of zero loci of proper transforms on  $Y'$  of  $(x_0 = 0)_{X'}$ ,  $(x_1 = 0)_{X'}$ ,  $(x_3 = 0)_{X'}$ ,  $(u = 0)_{X'}$  and  $(v = 0)_{X'}$  is empty. This implies that  $\psi$  is a morphism. The equations (2), (4) and (5) can be expressed as

$$M \cdot \begin{pmatrix} x_2 \\ w \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$M = \begin{pmatrix} u & c & h \\ -h & v & -u^2 + ue - hd \\ c & -ux_3 + ac & v - ub + cd + ah \end{pmatrix}.$$

We see that  $\det M$  is divisible by  $u$  and  $G := \det M/u$  is quadratic with respect to  $v$ . It follows that  $\psi$  is birational and the image of  $\psi$  is the weighted hypersurface  $Z'$  in  $\mathbb{P}(1, n, a_3, a_3 + 2n, 4a_2 - n)$  defined by the equation  $G = 0$ . Moreover, the projection to the coordinates  $x_0, x_1, x_3, u$  defines a double cover  $\pi: Z' \rightarrow \mathbb{P}(1, n, a_3, a_3 + 2n)$ .

We explain that the birational morphism  $\psi: Y' \rightarrow Z'$  is not an isomorphism. Let  $\tilde{S}$  be the proper transform on  $Y'$  of  $S := (c = h = u = 0) \subset X'$ . Here the section  $u$ , considered as a section on  $X'$ , is a polynomial in  $x_0, x_1, x_2, x_3$ , and  $w$ . We see that  $\psi(T) = (c = h = u = v = 0) \subset Z'$ . We have  $1 \leq \dim T \leq 2$  and  $\dim \psi(T) = \dim T - 1$ . This shows that  $\psi$  cannot be an isomorphism.

By [13, Lemma 3.2], either  $\varphi$  is not a maximal extraction or there is a birational involution of  $X'$  which is a Sarkisov link starting with  $\varphi$ .

We next treat the case where  $i \in \{38, 63\}$ . The standard polynomial of  $X' \subset \mathbb{P}(1, n, a_2, a_3, n)$  can be written as

$$F' = w^2 x_2 x_3 + w(x_3^2 a + x_3 b + c) + x_3^3 + x_3^2 d + x_3 e + h,$$

where  $a, b, \dots, e, h \in \mathbb{C}[x_0, x_1, x_2]$ . Note that  $a_2 + 2n = 2a_2$  and  $n = 3$  in this case. We have a symmetry between families  $\mathcal{G}'_i$  with  $i \in \{10, 26, 48\}$  and with  $i \in \{38, 63\}$ : the situation coincides after interchanging the role of  $x_2$  and  $x_3$ . Thus, by the

symmetric argument, we can construct the section  $u = w^2x_2 + w(x_3a+b) + x_3^2 + x_3d + e$  of degree  $a_2 + 2n$ , and the weighted blowup  $\varphi: Y' \rightarrow X'$  with

$$\text{wt}(x_0, x_1, x_2, x_3, u) = \frac{1}{n}(1, n, a_2, a_3 - n, a_2 + 2n),$$

which is a divisorial extraction (see Table 4 for the singularities of  $Y'$  along  $E$ ). Moreover, we have the anticanonical morphism  $\psi: Y' \rightarrow Z'$  whose base admits a double cover  $Z' \rightarrow \mathbb{P}(1, n, a_2, a_2 + 2n)$ . By [13, Lemma 3.2], either  $\varphi$  is not a maximal extraction or there is a birational involution of  $X'$  which is a Sarkisov link starting with  $\varphi$ . Therefore, we have the following.

**Theorem 6.2.** *Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in \{10, 26, 38, 48, 63\}$ ,  $\mathfrak{p} = \mathfrak{p}_4$  the  $cA/n$  point, and let  $\varphi_{(k,l)}: Y'_{(k,l)} \rightarrow X'$  an extremal divisorial extraction centered at  $\mathfrak{p}$  marked B.I. in the big table. Then, one of the following holds.*

- (1)  $\varphi_{(k,l)}$  is not a maximal extraction.
- (2) There is a birational involution of  $X'$  which is the Sarkisov link starting with  $\varphi_{(k,l)}$ .

*Proof.* If  $i \in \{38, 63\}$  (resp.  $i \in \{10, 26, 48\}$ ), then there is a unique extraction (resp. two extractions) marked B.I. The extraction  $\varphi: Y' \rightarrow X'$  corresponds to (one of)  $\varphi_{(k,l)}$  marked B.I. Thus the proof is completed for  $i \in \{38, 63\}$ . Suppose that  $i \in \{10, 26, 48\}$ . The proof is completed for  $\varphi = \varphi_{(a_2-n, a_3+2n)}$ . We need to consider the case where  $\varphi = \varphi_{(a_3+2n, a_2-n)}$ . If  $i = 10$ , then, by interchanging the role of  $x_2$  and  $x_3$  in the above construction, we obtain the  $(a_3 + 2n, a_2 - n)$ -blowup  $\varphi$ . If  $i = 26$  (resp. 48), then  $\varphi$  is obtained as the composite of  $\varphi_{(a_2-n, a_3+2n)}$  and the isomorphism  $\nu'$  (resp. the automorphism  $\mu$ ) defined in Definition 4.3 (resp. 4.2). Therefore, the proof for  $\varphi_{(a_3+2n, a_2-n)}$  follows from that for  $\varphi_{(a_2-n, a_3+2n)}$ .  $\square$

**6.2. Family  $\mathcal{G}'_{33}$  and  $(2, 7), (7, 2)$ -blowups.** Let  $X' = X'_{10} \subset \mathbb{P}(1, 1, 3, 5, 1)$  be a member of  $\mathcal{G}'_{33}$  and  $\mathfrak{p} = \mathfrak{p}_4$ . The aim of this subsection is to construct birational involutions starting with  $(2, 7)$ - and  $(7, 2)$ -blowups. This construction is a version of that of “invisible involutions” introduced in [3] (see also Section 5.2 and [13, Section 7]). We first give an explicit global description of  $(2, 7)$ - and  $(7, 2)$ -blowups  $\varphi: Y' \rightarrow X'$  and then construct a birational involution of a suitable model of  $Y'$ . The induced birational involution of  $Y'$  turns out to be a composite of inverse flips, flops and flips (in fact, it is a flop), so that it gives the desired Sarkisov link.

We can write the defining polynomial of  $X'$  as

$$(6) \quad \begin{aligned} F' = & w^2yz + w(zya_1 + za_4 + y^3 + y^2a_3 + ya_6 + a_9) \\ & + z^2 + zyb_2 + zb_5 + y^2b_4 + yb_7 + b_{10}, \end{aligned}$$

where  $a_i, b_i \in \mathbb{C}[x_0, x_1]$ . Filtering off terms divisible by  $wy$ , we obtain

$$(7) \quad F' = wyu + w(za_4 + a_9) + z^2 + zyb_2 + zb_5 + y^2b_4 + yb_7 + b_{10},$$

where

$$(8) \quad u := wz + za_1 + y^2 + ya_3 + a_6.$$

Multiplying  $F'$  by  $w$ , eliminating  $wz$  by the equation (8) and then filtering off terms divisible by  $y$ , we obtain

$$(9) \quad wF' = yv + w^2a_9 + (wa_4 + z + b_5)(u - za_1 - a_6) + wb_{10},$$

where

$$(10) \quad \begin{aligned} v := w^2 u - (wa_4 + z + b_5)(y + a_3) \\ + b_2(u - za_1 - y^2 - ya_3 - a_6) + wyb_4 + wb_7. \end{aligned}$$

Let  $U$  be the open subset subset of  $X'$  where  $w = 1$ . We see that  $U$  is naturally isomorphic to the subvariety of  $\mathbb{A}^6$  with affine coordinates  $x_0, x_1, y, z, u$  and  $v$  defined by three polynomials which are obtained by setting  $w = 1$  in (8), (9) and (10). Let  $\varphi: Y' \rightarrow X'$  be the weighted blowup of  $X'$  with

$$\text{wt}(x_0, x_1, y, z, u, v) = (1, 1, 2, 4, 6, 7)$$

and let  $E$  be the exceptional divisor of  $\varphi'$ .

**Lemma 6.3.** *The weighted blowup  $\varphi: Y' \rightarrow X'$  is a divisorial extraction centered at  $\mathfrak{p}$ .*

*Proof.* We have

$$E \cong (z + y^2 = yv + a_9 - (a_4 + z)za_1 = u - (a_4 + z)y - y^2b_2 + yb_4 = 0),$$

where the right-hand side is a WCI in  $\mathbb{P}(1_{x_0}, 1_{x_1}, 2_y, 4_z, 6_u, 7_v)$  and

$$J_\varphi = \begin{pmatrix} 0 & 0 & 2y & 1 & 0 & 0 & * \\ \frac{\partial a_9}{\partial x_0} + * & \frac{\partial a_9}{\partial x_1} + * & v & -a_1(2z + a_1) & 0 & y & b_{10} - a_4a_6 + * \\ * & * & b_4 + * & -y & 1 & 0 & -v + b_7 + * \end{pmatrix},$$

where  $*$  means a polynomial that is contained in the ideal  $(y, z)$ . By an explicit computation, we see that  $J_\psi$  is of rank 2 outside the set

$$\Sigma := \left( y = z = u = v = a_9 = \frac{\partial a_9}{\partial x_0} = \frac{\partial a_9}{\partial x_1} = b_{10} - a_4b_6 = 0 \right) \subset \mathbb{P}(1, 1, 2, 4, 6, 7).$$

We show that the system of equations

$$a_9 = \frac{\partial a_9}{\partial x_0} = \frac{\partial a_9}{\partial x_1} = b_{10} - a_4a_6 = 0$$

does not have a non-trivial solution, which will imply  $\Sigma = \emptyset$ . We assume that it has a non-trivial solution  $(x_0, x_1) = (\alpha_0, \alpha_1)$ . Set  $\alpha_i = a_i(\alpha_0, \alpha_1)$  for  $i = 4, 6, 9$  and  $\beta_{10} = b_{10}(\alpha_0, \alpha_1)$ . Let  $X \in \mathcal{G}_{33}$  be the birational counterpart of  $X'$ , which is defined by

$$\begin{aligned} F_1 &= ty + sz + (zya_1 + za_4 + y^3 + y^2a_3 + ya_6 + a_9) = 0, \\ F_2 &= ts - (z^2 + zyb_2 + zb_5 + y^2b_4 + yb_7 + b_{10}) = 0, \end{aligned}$$

in  $\mathbb{P}(1_{x_0}, 1_{x_1}, 3_y, 5_z, 4_s, 6_t)$ . Set  $\mathfrak{q} = (\alpha_0 : \alpha_1 : 0 : 0 : -\alpha_4 : -\alpha_6)$ . We have  $\mathfrak{q} \in X$  since  $\beta_{10} - \alpha_4\alpha_6 = \alpha_9 = 0$ . It is easy to verify that every partial derivative of  $F_1$  vanishes at  $\mathfrak{q}$ , which implies that  $X$  is not quasismooth. This is a contradiction.

It follows that  $\Sigma = \emptyset$  and thus  $Y'$  has only cyclic quotient singular points. It is then easy to see that  $Y'$  has singular points of type  $\frac{1}{2}(1, 1, 1)$  and  $\frac{1}{7}(1, 1, 6)$  at  $(0:0:1:-1:-1:0)$  and  $(0:0:0:0:0:1)$ , respectively. It follows that  $Y'$  has only terminal singularities and thus  $\varphi$  is a divisorial extraction.  $\square$

We see that  $\varphi = \varphi_{(2,7)}$  is a divisorial extraction which is a  $(2, 7)$ -blowup. Before going to the construction of birational involutions, we construct  $(7, 2)$ -blowup by taking the composite of  $(2, 7)$ -blowup and an automorphism of  $X'$ .

**Definition 6.4.** Let  $X'$  be a member of  $\mathcal{G}'_{33}$  with defining polynomial  $F' = w^2yz + w(zh_4 + h_9) + z^2 + zh_5 + h_{10}$ , where  $h_j \in \mathbb{C}[x_0, x_1, y]$ . We define  $\mu$  to be the automorphism of  $X'$  defined by the replacement  $z \mapsto -z - w^2y - wh_4 - h_5$ .

We see that  $\mu(\mathbf{p}) = \mathbf{p}$  and the composite  $\mu \circ \varphi_{(2,7)}$  is the  $(7, 2)$ -blowup.

We return to the case of  $(2, 7)$ -blowup  $\varphi = \varphi_{(2,7)}$ . Let  $\psi: W \rightarrow Y'$  be the Kawamata blowup of  $Y'$  at the  $\frac{1}{7}(1, 1, 6)$  point lying on  $E$  and let  $F \cong \mathbb{P}(1, 1, 6)$  be its exceptional divisor. For  $\lambda, \mu \in \mathbb{C}$ , we set  $S_\lambda := (x_1 - \lambda x_0 = 0)_{X'}$  and  $T_\mu := (u - \mu x_0^6 = 0)_{X'}$ . We see that  $S_\lambda$  is normal for a general  $\lambda \in \mathbb{C}$  and we define  $C_{\lambda, \mu}$  to be the scheme-theoretic intersection  $S_\lambda \cap T_\mu$ . Let  $\mathcal{M}$  be the linear system on  $X'$  generated by  $x_0^6, x_0^5 x_1, \dots, x_1^6$  and  $u$ , and let  $\mathcal{M}_W$  be its proper transform on  $W$ . We see that  $\mathcal{M}_W \subset |-6K_W|$  (in fact, equality holds) is base point free. Let  $\eta: W \rightarrow \mathbb{P}(1, 1, 6)$  be the morphism defined by  $\mathcal{M}_W$ , which resolves the indeterminacy of the projection  $\pi: X' \dashrightarrow \mathbb{P}(1_{x_0}, 1_{x_1}, 6_u)$ . For a curve of a divisor  $\Delta$  on  $X'$  or  $Y'$ , we denote by  $\hat{\Delta}$  the proper transform of  $\Delta$  on  $W$ . Note that  $\hat{C}_{\lambda, \mu}$  is the fiber of  $\eta$  over the point  $(1: \lambda: \mu)$ .

**Lemma 6.5.** *We have*

$$(\hat{E} \cdot \hat{C}_{\lambda, \mu}) = 3, (F \cdot \hat{C}_{\lambda, \mu}) = 1, (-K_W \cdot \hat{C}_{\lambda, \mu}) = 0.$$

*Proof.* Since  $K_{Y'} = \varphi^* K_{X'} + E$  and  $K_W = \psi^* K_{Y'} + \frac{1}{7}F$ , we have

$$\begin{aligned} (-K_{Y'}^3) &= (-K_{X'}^3) - (E^3) = \frac{2}{3} - \frac{9}{14} = \frac{1}{42}, \\ (-K_{W'}^3) &= (-K_{Y'}^3) - \frac{1}{7^3}(F^3) = \frac{1}{42} - \frac{1}{42} = 0. \end{aligned}$$

Since  $\tilde{S}_\lambda \sim_{\mathbb{Q}} -K_{Y'}$ ,  $\tilde{T}_\mu \sim_{\mathbb{Q}} -6K_{Y'}$  and  $\tilde{T}_\mu|_{\tilde{S}_\lambda} = \tilde{C}_{\lambda, \mu}$ , we have

$$(-K_{Y'} \cdot \tilde{C}_{\lambda, \mu}) = (-K_{Y'} \cdot \tilde{T}_\mu \cdot \tilde{S}_\lambda) = 6(-K_{Y'})^3 = \frac{1}{7},$$

which implies

$$(E \cdot \tilde{C}_{\lambda, \mu}) = (-K_{X'} \cdot \tilde{C}_{\lambda, \mu}) - (-K_{Y'} \cdot \tilde{C}_{\lambda, \mu}) = 4 - \frac{1}{7} = \frac{27}{7}.$$

Similarly, since  $\hat{S}_\lambda \sim_{\mathbb{Q}} -K_W$ ,  $\hat{T}_\mu \sim_{\mathbb{Q}} -6K_W$  and  $\hat{T}_\mu|_{\hat{S}_\lambda} = \hat{C}_{\lambda, \mu}$ , we have

$$(-K_W \cdot \hat{C}_{\lambda, \mu}) = (-K_W \cdot \hat{T}_\mu \cdot \hat{S}_\lambda) = 6(-K_W)^3 = 0,$$

which implies

$$(F \cdot \hat{C}_{\lambda, \mu}) = 7((-K_{Y'} \cdot \tilde{C}_{\lambda, \mu}) - (-K_W \cdot \hat{C}_{\lambda, \mu})) = 1.$$

Finally, we have  $\psi^* E = \hat{E} + \frac{6}{7}F$  and by taking the intersection number with  $\hat{C}_{\lambda, \mu}$ , we have  $(\hat{E} \cdot \hat{C}_{\lambda, \mu}) = 3$ .  $\square$

**Lemma 6.6.** *Suppose that, for a general  $\lambda \in \mathbb{C}$ , there is  $\mu \in \mathbb{C}$  (depending on  $\lambda$ ) such that  $C_{\lambda, \mu}$  is reducible. Then,  $\varphi$  is not a maximal extraction.*

*Proof.* Assume that  $C = C_{\lambda, \mu}$  is reducible. Then, there is a unique component  $C^\circ$  of  $C$  such that  $(G \cdot \hat{C}^\circ) = 1$ . Let  $C'$  be any component of  $C$  other than  $C^\circ$ . Then  $\hat{C}'$  is disjoint from  $G$  and we have  $(-K_{W'} \cdot \hat{C}') = 0$  since  $-K_{W'}$  is nef and  $(-K_{W'} \cdot \hat{C}) = 0$ . It follows that  $(-K_{Y'} \cdot \tilde{C}') = (-K_W \cdot \hat{C}') + \frac{1}{7}(F \cdot \hat{C}') = 0$ . We have  $(E \cdot \tilde{C}') = (K_{Y'} \cdot \tilde{C}') + (-K_{X'} \cdot C') > 0$ . This shows that there are infinitely many



curves on  $Y'$  which intersect  $-K_{W'}$  non-positively and  $E$  positively. By [13, Lemma 2.19],  $\varphi'$  is not a maximal singularity.  $\square$

**Theorem 6.7.** *Let  $X'$  be a member of  $\mathcal{G}'_{33}$  and let  $\varphi: Y' \rightarrow X'$  be a  $(2, 7)$ - or a  $(7, 2)$ -blowup centered at  $\mathfrak{p} = \mathfrak{p}_4$ . Then, either  $\varphi$  is not a maximal extraction, or there is a birational involution of  $X'$  that is a Sarkisov link starting with  $\varphi$ .*

*Proof.* We prove this for  $\varphi = \varphi_{(2,7)}$ . The proof for  $\varphi_{(7,2)}$  follows by composing  $\varphi_{(2,7)}$  with the automorphism  $\mu$ . The following argument is based on [3].

By Lemma 6.6, we may assume that, for a general  $\lambda \in \mathbb{C}$ ,  $C_{\lambda, \mu}$  is irreducible for every  $\mu \in \mathbb{C}$ . The morphism  $\eta: W \rightarrow \mathbb{P}(1, 1, 6)$  is an elliptic fibration and let  $\tau_W: W \dashrightarrow W$  be the birational involution defined as the reflection of the generic fiber with respect to the section  $F$ .

$$\begin{array}{ccccc}
 W & \xrightarrow{\tau_W} & W \\
 \psi \downarrow & \eta \swarrow & \nwarrow \eta & \downarrow \psi \\
 Y' & & Y' \\
 \varphi \downarrow & & \downarrow \varphi \\
 X' & \xrightarrow{\pi} & \mathbb{P}(1, 1, 6) & \xleftarrow{\pi} & X'
 \end{array}$$

We see that  $\tau_W$  is an isomorphism in codimension 1 since  $K_W$  is  $\eta$ -nef and it induces birational involutions  $\tau_{Y'}: Y' \dashrightarrow Y'$  and  $\tau: X' \dashrightarrow X'$ . Note that  $\tau_{Y'}$  is an isomorphism in codimension 1 since  $F$  is  $\tau_W$ -invariant.

We will show that  $\tau$  is not biregular. Assume to the contrary that  $\tau$  is biregular. We fix a general  $\lambda \in \mathbb{C}$  so that  $C_{\lambda, \mu}$  is irreducible for every  $\mu \in \mathbb{C}$ . The surface  $S_\lambda$  is  $\tau$ -invariant and  $\tau$  induces a biregular involution  $\tau_\lambda$  of  $S_\lambda$ , which induces a birational involution  $\hat{\tau}_\lambda$  of  $\hat{S}_\lambda$ . Note that  $\hat{\tau}_\lambda$  may not be biregular. Let  $\bar{S}_\lambda \rightarrow \hat{S}_\lambda$  be a composite of suitable blowups such that the birational involution  $\bar{\tau}_\lambda$  of  $\bar{S}_\lambda$  induced by  $\tau_\lambda$  is biregular. We denote by  $\sigma: \bar{S}_\lambda \rightarrow S_\lambda$  the composite of  $\bar{S}_\lambda \rightarrow \hat{S}_\lambda$  and  $\varphi|_{\hat{S}_\lambda}: \hat{S}_\lambda \rightarrow S_\lambda$  and by  $\bar{\pi}_\lambda: \bar{S}_\lambda \rightarrow \mathbb{P}^1$  the composite of  $\sigma$  and  $\pi|_{S_\lambda}: S_\lambda \dashrightarrow \mathbb{P}(1, 6) \cong \mathbb{P}^1$ . Let  $\bar{E}_\lambda$  and  $\bar{F}_\lambda$  be the proper transforms of  $\hat{E}|_{\hat{S}_\lambda}$  and  $F|_{\hat{S}_\lambda}$  on  $\bar{S}_\lambda$ , respectively, which are the prime  $\sigma$ -exceptional divisors that are not contracted by  $\bar{\pi}_\lambda$ . Denote by  $G_1, \dots, G_r$  the other prime  $\sigma$ -exceptional divisors.

Let  $\bar{C}_\lambda \subset \bar{S}_\lambda$  be the proper transform of a general fiber  $C_\lambda$  of  $\pi_\lambda$ . Since  $\bar{\tau}_\lambda|_{\bar{C}_\lambda}$  is the reflection with respect to the point  $\bar{F}_\lambda \cap \bar{C}_\lambda$  and  $\bar{E}_\lambda$  is  $\bar{\tau}_\lambda$ -invariant,  $(\bar{E}_\lambda - 3\bar{F}_\lambda)|_{\bar{C}_\lambda} \in \text{Pic}^0(\bar{C}_\lambda)$  is a 2-torsion. In particular,  $\bar{E}_\lambda - 3\bar{F}_\lambda$  is numerically equivalent to a linear combination of  $\bar{\pi}_\lambda$ -vertical divisors.

On the other hand, we have  $(x_0 = 0)|_{S_\lambda} = \Gamma$ , where

$$\Gamma = (x_0 = x_1 = w^2yz + wy^3 + z^2 = 0)$$

is an irreducible and reduced curve. We see that  $\Gamma$  and  $\{C_{\lambda, \mu} \mid \mu \in \mathbb{C}\}$  are all the fibers of  $\pi_\lambda$ . Since  $C_{\lambda, \mu}$  is irreducible for every  $\mu \in \mathbb{C}$  and all the fibers of  $\bar{\pi}_\lambda$  are numerically equivalent to each other, we have

$$\bar{E}_\lambda - 3\bar{F}_\lambda \sim_{\mathbb{Q}} \gamma \bar{\Gamma} + \sum_{i=1}^r c_i G_i,$$

for some  $\gamma, c_1, \dots, c_r \in \mathbb{C}$ , where  $\bar{\Gamma}$  is the proper transform of  $\Gamma$ . We see that  $\gamma \neq 0$  since the curves  $\bar{E}_\lambda, \bar{F}_\lambda, G_1, \dots, G_r$  are  $\sigma$ -exceptional and their intersection form is

negative-definite. This shows  $\Gamma \sim_{\mathbb{Q}} 0$ , which is a contradiction since  $(A \cdot \Gamma) \neq 0$ . Therefore,  $\tau: X' \dashrightarrow X'$  is not biregular and, by [13, Lemma 2.24],  $\tau$  is a Sarkisov link starting with  $\varphi$ .  $\square$

**Remark 6.8.** It is straightforward to see that  $-K_{Y'}$  is nef and  $\tau_{Y'}: Y' \dashrightarrow Y'$  is a flop. The anticanonical model  $Z$  of  $Y'$  is a (non- $\mathbb{Q}$ -factorial) Fano 3-fold with degree  $1/42$ . By looking at the Fletcher's list [7], it is quite likely that  $Z$  is a weighted hypersurface  $Z = Z_{42} \subset \mathbb{P}(1, 1, 6, 14, 21)$  of degree 42. If one can find sections of degree 14 and 21 on  $X'$  that lift to plurianticanonical sections on  $Y'$ , then we can construct  $Z$  explicitly as in the argument of the previous subsection, and the existence of flop will follow from this.

## 7. EXCLUSION OF NONSINGULAR POINTS

The aim of this section is to exclude nonsingular points as maximal center. Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ .

**Definition 7.1** ([5, Definition 5.2.4]). Let  $\mathfrak{p} \in X'$  be a point. We say that a Weil divisor class  $L$  on  $X'$  *isolates*  $\mathfrak{p}$  if  $\mathfrak{p}$  is an isolated component of the linear system

$$\mathcal{L}_{\mathfrak{p}}^s := |\mathcal{I}_{\mathfrak{p}}(sL)|$$

for some integer  $s > 0$ .

**Lemma 7.2** ([5]). *Let  $\mathfrak{p} \in X'$  be a nonsingular point. If  $lA$  isolates  $\mathfrak{p}$  for some  $0 < l \leq 4/(A^3)$ , then  $\mathfrak{p}$  is not a maximal center.*

*Proof.* See [5, Proof of (A)].  $\square$

Let  $\mathbb{P}(a_0, \dots, a_4)$  be the ambient WPS of  $X'$ . For  $j, m = 0, \dots, 4$  with  $j \neq m$ , we define

$$\tilde{a}_j := \max_{0 \leq k \leq 4, k \neq j} \text{lcm}(a_j, a_k), \quad \tilde{a}_{j;m} := \max_{0 \leq k \leq 4, k \neq j, m} \text{lcm}\{a_j, a_k\}.$$

For  $m = 0, \dots, 4$ , we denote by  $\pi_m$  the restriction of the projection from  $\mathfrak{p}_m$  to  $X'$  and by  $\text{Exc}(\pi_m) \subset X'$  the locus contracted by  $\pi_m$ .

**Proposition 7.3** ([13, Proposition 5.1]). *Let  $\mathfrak{p} = (\xi_0 : \dots : \xi_4)$  be a nonsingular point of  $X'$ . Then, the following assertions hold.*

- (1) *If  $\xi_j \neq 0$  then  $\tilde{a}_j A$  isolates  $\mathfrak{p}$ .*
- (2) *If  $\xi_j \neq 0$  and  $\mathfrak{p} \notin \text{Exc}(\pi_m)$  for some  $m \neq j$ , then  $\tilde{a}_{j;m} A$  isolates  $\mathfrak{p}$ .*

**Theorem 7.4.** *Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA}^* \cup I_{cA/n} \cup I_{cD/3}$ . Then no nonsingular point of  $X'$  is a maximal center.*

*Proof.* Let  $\mathbb{P}(a_0, \dots, a_4)$  be the ambient WPS of  $X'$  as above and let  $F'$  be the defining polynomial of  $X'$ . Suppose  $i \in \{7, 10, 18, 21, 22, 36, 38, 44, 52, 57, 62, 63\}$ . Then the inequality

$$\tilde{a} := \max_{0 \leq k < l \leq 4} \text{lcm}(a_j, a_k) \leq 4/(A^3)$$

holds. Note that  $\tilde{a}_j \leq \tilde{a} \leq 4/(A^3)$  for any  $j$ . Thus Lemma 7.2 and Proposition 7.3 imply that no nonsingular point is a maximal center.

Suppose that  $i \in \{26, 28, 33, 48, 61\}$ . Then, there is  $0 \leq m \leq 4$  such that  $x_m^e \in F'$  for some  $e > 0$  and the inequality

$$b_m := \max_{0 \leq k < l \leq 4, k, l \neq m} \text{lcm}(a_k, a_l) \leq 4/(A^3)$$

holds. The assertion  $x_m^e \in F'$  implies  $\text{Exc}(\pi_m) = \emptyset$  and we have  $\tilde{a}_{j;m} \leq b_m$  for any  $j \neq m$ . Thus Lemma 7.2 and Proposition 7.3 imply that non nonsingular point is a maximal center.

Let  $X' = X'_7 \subset \mathbb{P}(1, 1, 2, 3, 1)$  be a member of  $\mathcal{G}'_{16}$  and let  $\mathbf{p} = (\xi_0 : \xi_1 : v : \zeta : \omega) \in X'$  be a nonsingular point. By the generality condition,  $y^2 z \in F'$ . This implies that  $(x_0 = x_1 = w = 0)_{X'}$  consists of singular points and hence  $\mathbf{p}$  is contained in one of the open subsets  $(x_0 \neq 0)$ ,  $(x_1 \neq 0)$  and  $(w \neq 0)$ . By Proposition 7.3,  $3A$  isolates  $\mathbf{p}$  and thus  $\mathbf{p}$  is not a maximal center since  $3 < 4/(A^3) = 24/7$ .

Finally, let  $X' = X'_5 \subset \mathbb{P}(1, 1, 1, 2, 1)$  be a member of  $\mathcal{G}'_6$  and  $\mathbf{p} = (\xi_0 : \xi_1 : \xi_2 : v : \omega)$  a nonsingular point of  $X$ . If  $\mathbf{p} \notin \text{Exc}(\pi_3)$ , then  $A$  isolates  $\mathbf{p}$  by Proposition 7.3. It follows that  $\mathbf{p} \notin \text{Exc}(\pi_3)$  is not a maximal center since  $4/(A^3) = 8/5 > 1$ . If  $\mathbf{p} \in \text{Exc}(\pi_3)$ , then the proof [5, Proof of (B) in Section 5.3] works for our case and we have that  $\mathbf{p}$  is not a maximal center in this case.  $\square$

## 8. EXCLUSION OF QUOTIENT SINGULAR POINTS

Throughout this section, let  $\mathbf{p}$  be a terminal quotient singular point of  $X' \in \mathcal{G}'_i$  with empty third column in the big table. The aim of this section is to exclude  $\mathbf{p}$  as a maximal center. We denote by  $\varphi': Y' \rightarrow X'$  the Kawamata blow up at  $\mathbf{p}$  with the exceptional divisor  $E$  and put  $B = -K_{Y'}$  as usual. For a divisor or a curve  $\Delta$  on  $X'$ , we denote by  $\tilde{\Delta}$  the proper transform of  $\Delta$  on  $Y'$ .

We first treat the points such that a set of polynomials and a divisor of the form  $bB + eE$  are given in the second column of the table. We note that  $(B^3) \leq 0$  in this case. Let  $\Lambda$  be the set of polynomials in the second column. A coordinate with a prime means that it is the tangent coordinate of  $X'$  at  $\mathbf{p}$ . For example, let  $X' = X'_9 \subset \mathbb{P}(1, 1, 2, 3, 3)$  be a member of  $\mathcal{G}'_{21}$  and  $\mathbf{p}$  the point of type  $\frac{1}{2}(1, 1, 1)$ . We see that  $y^3 \in f_6$  in the defining polynomial  $w^2 x_0 y + w f_6 + g_9$  because otherwise  $X'$  does not contain a singular point of type  $\frac{1}{2}(1, 1, 1)$ . Then we have  $w' = w + (\text{other terms})$ .

**Lemma 8.1.** *Let  $X'$  be a member of  $\mathcal{G}'_i$  and  $\mathbf{p} \in X'$  a terminal quotient singular point marked a set of polynomials together with a divisor of the form  $N := bB + eE$  in the third column of the big table. Then, the divisor  $N$  is nef and  $(N \cdot B^2) \leq 0$ . In particular,  $\mathbf{p}$  is not a maximal center.*

*Proof.* Let  $\Lambda = \{h_1, \dots, h_l\}$  be the set of polynomials in the second table. It is straightforward to see that  $(h_1 = \dots = h_l)_{X'}$  is a finite set of points including  $\mathbf{p}$  and we omit the proof (see Example 8.2). Suppose that  $h_j$  vanishes along  $E$  to order  $c_j/r$ , where  $r$  is the index of  $\mathbf{p}$ , and set  $b_j = \deg h_j$ . Then the proper transform of  $(h_j = 0)_{X'}$  on  $Y'$  defines the divisor  $N_j \sim_{\mathbb{Q}} b_j B + e_j E$ , where  $e_j = (b_j - c_j)/r \geq 0$ . Let  $k$  be an index such that

$$e_k/b_k = \max_{1 \leq j \leq l} \{e_j/b_j\}.$$

Then  $N_k$  is the divisor  $N = bB + eE$  given in the second column and we observe that the inequality  $b > re$  holds. By [13, Lemma 6.6],  $N = N_k$  is nef. The verification of  $(N \cdot B^2) \leq 0$  is straightforward. Therefore, by [13, Corollary 2.16],  $\mathbf{p}$  is not a maximal center.  $\square$

**Example 8.2.** Let  $X' = X'_{15} \subset \mathbb{P}(1, 2, 3, 5, 5)$  be a member of  $\mathcal{G}'_{57}$  with defining polynomial  $F' = w^2 z t + w f_{12} + g_{14}$  and  $\mathbf{p}$  a point of type  $\frac{1}{2}(1, 1, 1)$ . Note that if  $y^6 \notin f_{12}$ , then there is no  $\frac{1}{2}(1, 1, 1)$  point on  $X'$ . We assume  $y^6 \in f_6$ . After replacing

$w$ , we may assume that there is no monomial in  $w$  divisible by  $y^6$  in  $g_{14}$ . In this setting, we have  $w' = w$ . Since we have  $F'(0, y, 0, t, 0) = \alpha t^2$  for some  $\alpha \neq 0$ , we see that  $(x = z = w = 0)_{X'} = \{\mathbf{p}\}$ .

Since we can choose  $x, z, t$  as local orbifold coordinates of  $X'$  at  $\mathbf{p}$ , they vanish along  $E$  to order  $1/2$ . It is clear that  $w$  vanishes along  $E$  to order at least  $2/2$  since  $(w = 0)_{X'}$  is Cartier along  $\mathbf{p}$ . Thus, the proper transforms of  $(x = 0)_{X'}$ ,  $(z = 0)_{X'}$  and  $(w = 0)_{X'}$  defines divisors  $B$ ,  $3B + E$  and  $2B$ , respectively. It follows that  $N = 3B + E$  and the inequality  $b > re$  holds since  $b = 3$ ,  $e = 1$  and  $r = 2$ . Finally, we have

$$(B^2 \cdot 3B + E) = 3(A^3) - \frac{1}{2^3}(E^3) = \frac{1}{10} - \frac{1}{2} < 0.$$

This completes all the computations for  $X' \in \mathcal{G}'_{57}$  required in the proof of Lemma 8.1.

**Lemma 8.3.** *Let  $X' = X'_9 \subset \mathbb{P}(1, 1, 2, 3, 3)$  be a member of  $\mathcal{G}'_{21}$ . Then the singular point  $\mathbf{p} = \mathbf{p}_2$  of type  $\frac{1}{2}(1, 1, 1)$  is not a maximal center.*

*Proof.* The defining polynomial of  $X'$  is of the form  $F' = w^2 x_0 y + w f_6 + g_9$ . Since  $z^2 \in f_6$ , we may assume that  $z^3 \notin g_9$  and the coefficient of  $z^2$  in  $f_6$  is 1 after replacing  $w$  with  $w + \eta z$  for a suitable  $\eta \in \mathbb{C}$  and then re-scaling  $z$ . We write  $f_6(0, 0, y, z) = z^2 + \alpha y^3$  and  $g_9(0, 0, y, z) = \beta z y^3$  for some  $\alpha, \beta \in \mathbb{C}$ . Note that neither  $\alpha$  nor  $\beta$  is zero by the generality condition.

We see that  $x_0$  and  $x_1$  vanish along  $E$  to order  $1/2$ . Let  $S, T$  be general members of the pencil  $|A|$ . Then  $\tilde{S} \cap \tilde{T}$  is the proper transform of the curve

$$(x_0 = x_1 = 0)_{X'} = (x_0 = x_1 = w(z^2 + \alpha y^3) + \beta z y^3 = 0),$$

which is irreducible and reduced since  $\alpha \neq 0$  and  $\beta \neq 0$ . Thus, by [13, Lemma 2.17],  $\tilde{S} \cap \tilde{T}$  is not a maximal center since  $\tilde{S}, \tilde{T} \sim_{\mathbb{Q}} B$  and  $(B \cdot \tilde{S} \cdot \tilde{T}) = (B^3) = 0$ .  $\square$

**Lemma 8.4.** *Let  $X' = X'_{12} \subset \mathbb{P}(1, 2, 3, 5, 2)$  be a member of  $\mathcal{G}'_{44}$ . Then a singular point  $\mathbf{p} \in X'$  of type  $\frac{1}{2}(1, 1, 1)$  is not a maximal center.*

*Proof.* The defining polynomial of  $X'$  is of the form  $F' = w^2 z t + w f_{10} + g_{12}$ . We may assume  $y^5 \in f_{10}$  because otherwise  $X'$  does not contain a point of type  $\frac{1}{2}(1, 1, 1)$ . After replacing  $w$ , we may assume that there is no monomial divisible by  $y^5$  in  $g_{12}$ . We may moreover assume that the coefficient of  $z^4$  in  $g_{12}$  is 1 after re-scaling  $z$ . Let  $\alpha, \beta$  and  $\gamma$  be the coefficients of  $y^3 z^2$ ,  $y^2 z t$  and  $y t^2$  in  $F'$ , respectively. We see that  $x, z, t$  vanish along  $E$  to order  $1/2$ . By looking at the defining equation, the section  $w$  vanishes along  $E$  to order  $2/2$ . We define  $S := (x = 0)_{X'}$  and  $T := (w + \delta x^2 = 0)_{X'}$  for a general  $\delta \in \mathbb{C}$ . Then  $\tilde{S} \sim_{\mathbb{Q}} B$  and  $\tilde{T} \sim_{\mathbb{Q}} 2B$ . We have

$$S \cap T = (x = w = 0)_{X'} = (x = w = \alpha y^3 z^2 + \beta y^2 z t + \gamma y t^2 + z^4 = 0).$$

If  $\tilde{S} \cap \tilde{T}$  is irreducible (possibly non-reduced), then, by [13, Lemma 2.17],  $\mathbf{p}$  is not a maximal center since  $(B \cdot \tilde{S} \cdot \tilde{T}) = 2(B^3) < 0$ .

We continue the proof assuming that  $\tilde{S} \cap \tilde{T}$  is reducible, which is equivalent to the condition  $\gamma = 0$  and  $(\alpha, \beta) \neq (0, 0)$ . Assume first  $\gamma = 0$  and  $\beta \neq 0$ . Then  $S|_T = \Gamma + \Delta$ , where  $\Gamma = (x = w = z = 0)$  and  $\Delta = (x = w = \alpha y^3 z + \beta y^2 t + z^3 = 0)$ . We have  $(E \cdot \tilde{\Gamma}) = 1$  so that

$$(B \cdot \tilde{\Gamma}) = (A \cdot \Gamma) - \frac{1}{2}(E \cdot \tilde{\Gamma}) = \frac{1}{10} - \frac{1}{2} = -\frac{2}{5}.$$

Since  $\tilde{S} \sim_{\mathbb{Q}} B$ ,  $\tilde{T} \sim_{\mathbb{Q}} 2B$  and  $\tilde{S}|_{\tilde{T}} = \tilde{\Gamma} + \tilde{\Delta}$ , we compute

$$-\frac{2}{5} = (\tilde{S}|_{\tilde{T}} \cdot \tilde{\Gamma})_{\tilde{T}} = (\tilde{\Gamma}^2)_{\tilde{T}} + (\tilde{\Gamma} \cdot \tilde{\Delta})_{\tilde{T}}$$

and

$$-\frac{3}{5} = 2(B^3) = (\tilde{S}|_{\tilde{T}}^2)_{\tilde{T}} = (\tilde{\Gamma})_{\tilde{T}}^2 + 2(\tilde{\Gamma} \cdot \tilde{\Delta})_{\tilde{T}} + (\tilde{\Delta}^2)_{\tilde{T}}.$$

Set  $m := (\tilde{\Gamma} \cdot \tilde{\Delta})_{\tilde{T}} > 0$ . By the above displayed equations, the intersection matrix

$$\begin{pmatrix} (\tilde{\Gamma}^2) & (\tilde{\Gamma} \cdot \tilde{\Delta}) \\ (\tilde{\Gamma} \cdot \tilde{\Delta}) & (\tilde{\Delta}^2) \end{pmatrix} = \begin{pmatrix} -2/5 - m & m \\ m & -1/5 - m \end{pmatrix}$$

is negative-definite. Thus, by [13, Lemma 2.18],  $\mathfrak{p}$  is not a maximal center.

Assume  $\gamma = \beta = 0$ . Note that  $\alpha \neq 0$  in this case. Filtering off terms divisible by  $y^3$ , we write  $F = y^3G + H$ . We set  $G_{\lambda} = G(x, y, z, t, \lambda x^2)$  and  $H_{\lambda} = H(x, y, z, t, \lambda x^2)$ . Note that the coefficients of  $z^2$  and  $z^4$  in  $G$  and  $H$  are  $\alpha \neq 0$  and 1, respectively, and let  $\varepsilon$  be the coefficient of  $tz^2x$  in  $H$ . By eliminating  $z^2$  in  $H_{\lambda}$  in terms of  $G_{\lambda} = 0$ , we have

$$(w - \lambda x^2 = G_{\lambda} = H_{\lambda} = 0) = (w - \lambda x^2 = G_{\lambda} = x^2 P_{\lambda} = 0) \subset X',$$

where  $P_{\lambda} \in \mathbb{C}[x, y, z, t]$ . We set  $C_{\lambda} := (w - \lambda x^2 = G_{\lambda} = P_{\lambda} = 0)$ . We see that  $\tilde{C}_{\lambda}$  intersect  $E \cong \mathbb{P}^2$  at 4 points so that  $(E \cdot \tilde{C}_{\lambda}) = 4$ . Hence, we have

$$(-K_{Y'} \cdot \tilde{C}_{\lambda}) = (K_X \cdot C_{\lambda}) - \frac{1}{2}(E \cdot \tilde{C}_{\lambda}) = 0.$$

If  $\tilde{C}_{\lambda}$  is reducible, then there is a component of  $\tilde{C}_{\lambda}$  which intersects  $-K_{Y'}$  non-positively and  $E$  positively since there is at least one component that intersects  $E$ , and since a component of  $C_{\lambda}$  that is disjoint from  $E$  intersects  $-K_{Y'}$  positively. It follows that there are infinitely many irreducible curves on  $Y'$  which intersects  $-K_{Y'}$  non-positively and  $E$  positively. By [13, Lemma 2.19],  $\mathfrak{p}$  is not a maximal center.  $\square$

**Lemma 8.5.** *Let  $X' = X'_{15} \subset \mathbb{P}(1, 3, 4, 5, 3)$  be a member of  $\mathcal{G}'_{62}$  or  $\mathcal{G}'_{63}$ . Then a singular point  $\mathfrak{p} \in X'$  of type  $\frac{1}{3}(1, 1, 2)$  is not a maximal center.*

*Proof.* If  $X' \in \mathcal{G}'_{62}$  (resp.  $\mathcal{G}'_{63}$ ), then the defining polynomial of  $X'$  is of the form  $F' = w^3y^2 + w^2yf_6 + wf_{12} + g_{15}$  (resp.  $F' = w^2zt + wf_{12} + g_{15}$ ). If  $X' \in \mathcal{G}'_{63}$ , then we may assume  $y^4 \in f_{12}$  because otherwise  $X'$  does not contain a point of type  $\frac{1}{3}(1, 1, 2)$ . After replacing  $w$ , we may assume that there is no monomial divisible by  $y^4$  in  $g_{15}$ . Let  $\alpha, \beta$  and  $\gamma$  be the coefficients of  $y^2zt$ ,  $yz^3$  and  $t^3$  in  $F'$ , respectively. By quasismoothness of  $X'$ , we have  $\gamma \neq 0$ . We see that  $x, z, t$  vanish along  $E$  to order  $1/3, 1/3$  and  $2/3$ , respectively. By looking at the defining equation  $F' = 0$ , we see that  $w$  vanishes along  $E$  to order  $3/3$ . We define  $S = (x = 0)_{X'}$  and  $T = (w + \delta x^3 = 0)_{X'}$  for a general  $\delta \in \mathbb{C}$ . Then  $\tilde{S} \sim_{\mathbb{Q}} B$  and  $\tilde{T} \sim_{\mathbb{Q}} 3B$ . We have

$$S \cap T = (x = w = 0)_{X'} = (x = w = \alpha y^2zt + \beta yz^3 + \gamma t^3 = 0).$$

If  $\tilde{S} \cap \tilde{T}$  is irreducible (possibly non-reduced), then, by [13, Lemma 2.17],  $\mathfrak{p}$  is not a maximal center since  $(B \cdot \tilde{S} \cdot \tilde{T}) = 3(B^3) < 0$ .

We assume that  $S \cap T$  is reducible, which is equivalent to the condition  $\beta = 0$  and  $\alpha \neq 0$ . We define  $\Gamma := (x = w = z = 0)$  and  $\Delta := (x = w = \alpha y^2z + \gamma t^2 = 0)$

so that  $S|_T = \Gamma + \Delta$ . Note that we have  $\tilde{S}|_{\tilde{T}} = \tilde{\Gamma} + \tilde{\Delta}$ . We see that  $\tilde{\Gamma}$  intersects  $E$  transversally at a single nonsingular point so that  $(E \cdot \tilde{\Gamma}) = 1$ . Hence,

$$(B \cdot \tilde{\Gamma}) = (A \cdot \Gamma) - \frac{1}{3}(E \cdot \tilde{\Gamma}) = -\frac{1}{4}.$$

Now we compute

$$-\frac{1}{4} = (\tilde{S}|_{\tilde{T}} \cdot \tilde{\Gamma})_{\tilde{T}} = (\tilde{\Gamma}^2)_{\tilde{T}} + (\tilde{\Gamma} \cdot \tilde{\Delta})_{\tilde{T}},$$

and

$$-\frac{1}{4} = 3(B^3) = (B|_T)^2_{\tilde{T}} = (\tilde{\Gamma} + \tilde{\Delta})^2_{\tilde{T}} = (\tilde{\Gamma}^2)_{\tilde{T}} + 2(\tilde{\Gamma} \cdot \tilde{\Delta})_{\tilde{T}} + (\tilde{\Delta}^2)_{\tilde{T}}.$$

Set  $m := (\tilde{\Gamma} \cdot \tilde{\Delta}) > 0$ . By the above displayed equations, we see that the intersection matrix

$$\begin{pmatrix} (\tilde{\Gamma}^2) & (\tilde{\Gamma} \cdot \tilde{\Delta}) \\ (\tilde{\Gamma} \cdot \tilde{\Delta}) & (\tilde{\Delta}^2) \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} - m & m \\ m & -m \end{pmatrix}$$

is negative-definite. Therefore, by Lemma [13, Lemma 2.18],  $\mathfrak{p}$  is not a maximal center.  $\square$

**Lemma 8.6.** *Let  $X' = X'_{10} \subset \mathbb{P}(1, 1, 3, 5, 1)$  be a member of  $\mathcal{G}'_{33}$ . Then, a singular point  $\mathfrak{p}$  of type  $\frac{1}{3}(1, 1, 2)$  is not a maximal center.*

*Proof.* The defining polynomial of  $X'$  is of the form  $w^2yz + wf_9 + g_{10}$ . Since  $y^3 \in f_9$ , we may assume that there is no monomial divisible by  $y^3$  in  $g_{10}$  after replacing  $w$ . We see that  $x_0, x_1, z, w$  vanish along  $E$  to order respectively  $1/3, 1/3, 2/3, 4/3$ . We set  $S := (w = 0)_{X'}$  and  $T := (x_0 = 0)_{X'}$ . Then  $\tilde{S} \sim_{\mathbb{Q}} B - E$  and  $\tilde{T} \sim_{\mathbb{Q}} B$ . We have

$$S \cap T = (x_0 = w = 0)_{X'} = (x_0 = w = g_{10}(0, x_1, y, z) = 0).$$

Note that  $z^2 \in g_{10}$ . It follows that if  $S \cap T$  is reducible, then it is the union of two curves  $\Gamma_1$  and  $\Gamma_2$ , where  $\Gamma_i := (x_0 = w = \alpha_i z + \beta_i y x_1^2 + \gamma_i x_1^5 = 0)$  for some  $\alpha_i \neq 0, \beta_i, \gamma_i \in \mathbb{C}$  for  $i = 1, 2$ . We have  $\tilde{S} \cap \tilde{T} = \tilde{\Gamma}_1 + \tilde{\Gamma}_2$ . We see that  $\tilde{\Gamma}_1$  is numerically equivalent to  $\tilde{\Gamma}_2$  since  $(\varphi^* A \cdot \tilde{\Gamma}_1) = (\varphi^* A \cdot \tilde{\Gamma}_2) = 1/3$  and  $(E \cdot \tilde{\Gamma}_1) = (E \cdot \tilde{\Gamma}_2) = 1$ . Thus the support of  $\tilde{\Gamma} := \tilde{S} \cap \tilde{T}$  is either irreducible or it is the union of two curves which are numerically equivalent to each other. We have

$$(\tilde{T} \cdot \tilde{\Gamma}) = (\tilde{T}^2 \cdot \tilde{S}) = (A^3) - \frac{4}{3^3}(E^3) = \frac{2}{3} - \frac{2}{3} = 0.$$

By [13, Lemma 2.17],  $\mathfrak{p}$  is not a maximal center.  $\square$

The following is the conclusion of this section.

**Theorem 8.7.** *Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ . Then, no terminal quotient singular point with empty third column in the big table is a maximal center.*

*Proof.* This follows from Lemmas 8.1, 8.3, 8.4, 8.5 and 8.6.  $\square$

## 9. EXCLUSION OF DIVISORIAL EXTRACTIONS CENTERED AT $cA/n$ POINTS

Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^*$  and  $\mathfrak{p} = \mathfrak{p}_4 \in X'$  the  $cA/n$  point. The aim of this section is to show that no divisorial extraction  $\varphi': Y' \rightarrow X'$  centered at  $\mathfrak{p}$  with  $(-K_{Y'})^3 \leq 0$  is a maximal extraction.

Let  $\varphi': Y' \rightarrow X'$  be a divisorial extraction centered at  $\mathfrak{p}$  such that  $(B^3) \leq 0$ , where  $B := -K_{Y'}$ . Such an extraction is the one marked “none” in the big table. Suppose  $i \in \{16, 22, 26, 33, 48\}$ . We define  $S := (x_0 = 0)_{X'}$  and  $T := (x_1 = 0)_{X'}$  if  $i \in$

TABLE 5. Defining equation of  $\Gamma$

No.	Equations	Conditions
16	$w^2yz + w(\alpha y^3 + \beta z^2) + \gamma y^2z = 0$	$\alpha, \beta, \gamma \neq 0$
22	$w^2yz + \alpha y^4 + \beta y^2z + z^2$	$\alpha \neq 0$
26	$w^2yz + \alpha wz^2 + z^3 = 0$	$\alpha \neq 0$
33	$w^2yz + \alpha wy^3 + z^2 = 0$	$\alpha \neq 0$
44	$w^2zt + \alpha wt^2 + y^3 = 0$	$\alpha \neq 0$
48	$w^2yz + y^3 + z^2 = 0$	
57	$w^2zt + \alpha wz^4 + t^2 = 0$	$\alpha \neq 0$
63	$w^2zt + \alpha wz^3 + t^3 = 0$	$\alpha \neq 0$

$\{16, 22, 26, 33, 48\}$ , and define  $S := (x = 0)_{X'}$  and  $T := (y = 0)_{X'}$  if  $i \in \{44, 57, 63\}$ . We set  $m = 1$  if  $i \in \{16, 22, 26, 33, 48\}$ , and  $m = \deg y$  if  $i \in \{44, 57, 63\}$ . We have  $S \sim_{\mathbb{Q}} A$  and  $T \sim_{\mathbb{Q}} mA$ . For a divisor or a curve  $\Delta$  on  $X'$ , we denote by  $\tilde{\Delta}$  the proper transform of  $\Delta$  on  $Y'$ .

**Lemma 9.1.** *The scheme-theoretic intersection  $\Gamma := S \cap T$  is an irreducible and reduced curve. Moreover, we have  $\tilde{S} \sim_{\mathbb{Q}} B$ ,  $\tilde{T} \sim_{\mathbb{Q}} mB$  and  $\tilde{S} \cap \tilde{T} = \tilde{\Gamma}$ .*

*Proof.* If  $i \in \{16, 22, 26, 33, 48\}$  (resp.  $i \in \{44, 57, 63\}$ ), then  $(x_0 = x_1 = 0)$  (resp.  $(x = y = 0)$ ) is a weighted projective plane, which we denote by  $\mathbb{P}$ , and  $\Gamma$  is isomorphic to the hypersurface in  $\mathbb{P}$  defined by the equation given in Table 9. The conditions on  $\alpha, \beta, \gamma$  in the table are satisfied by the generality condition imposed on the family  $\mathcal{G}'_i$ . Thus, by a straightforward argument,  $\Gamma$  is irreducible and reduced.

We see that  $x_0$  and  $x_1$  (resp.  $x$  and  $y$ ) vanish along  $E$  to order  $1/n$  and  $1/n$  (resp.  $1/n$  and  $m/n$ ). This implies  $\tilde{S} \sim_{\mathbb{Q}} B$  and  $\tilde{T} \sim_{\mathbb{Q}} mB$ . Moreover,  $\tilde{S} \cap \tilde{T} \cap E$  consists of two points and, in particular, does not contain a curve. This shows  $\tilde{S} \cap \tilde{T} = \tilde{\Gamma}$ .  $\square$

**Theorem 9.2.** *Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA}^* \cup I_{cA/n}$  and  $\mathbf{p} = \mathbf{p}_4 \in X'$  the  $cA$  or  $cA/n$  point. Then no extremal extraction  $\varphi': Y' \rightarrow X'$  with  $(-K_{Y'})^3 \leq 0$  is a maximal center.*

*Proof.* By Lemma 9.1, we have  $\tilde{S} \sim_{\mathbb{Q}} B$ ,  $\tilde{T} \sim_{\mathbb{Q}} mB$  and  $\tilde{S} \cap \tilde{T} = \tilde{\Gamma}$  is an irreducible and reduced curve. It follows that

$$(\tilde{T} \cdot \tilde{\Gamma}) = (\tilde{T} \cdot \tilde{S} \cdot \tilde{T}) = m^2(B^3) \leq 0.$$

Therefore, by [13, Lemma 2.17],  $\varphi'$  is not a maximal extraction.  $\square$

## 10. EXCLUSION OF CURVES

**10.1. Exclusion of most of the curves.** We can exclude most of the curves as follows.

**Lemma 10.1.** *Let  $X'$  be a member of the family  $\mathcal{G}'_i$  and  $\Gamma \subset X'$  an irreducible and reduced curve. Then,  $\Gamma$  is not a maximal center except possibly for the following cases.*

- No. 6 and  $\deg \Gamma = 1, 2$ .
- No. 7 and  $\deg \Gamma = \frac{1}{2}, 1$ .
- No. 9, 10, 16 and  $\deg \Gamma = 1$ .
- No. 18 and  $\deg \Gamma = \frac{1}{2}$ .



- No. 21 and  $\deg \Gamma = \frac{1}{3}$ .

Here, in any of the above exceptions,  $\Gamma$  passes through the  $cA/n$  point  $p_4$  and does not pass through any terminal quotient singular point.

*Proof.* We may assume that  $\Gamma$  does not pass through a terminal quotient singular point since there is no divisorial extraction centered along a curve through such a point (see [10]). We see  $n \deg \Gamma = (nA \cdot \Gamma) \in \mathbb{Z}_{>0}$ , where  $n$  is the index of the singularity  $(X', p_4)$ . Thus,  $\deg \Gamma \in \frac{1}{n}\mathbb{Z}_{>0}$ . By [5, Proof of Theorem 5.1.1],  $\Gamma$  is not a maximal center if  $\deg \Gamma \geq (A^3)$ . By checking each family individually, a curve  $\Gamma$  on  $X'$  such that  $\deg \Gamma < (A^3)$  is one of the curves listed in the statement. Finally, let  $\Gamma$  be one of the curves in the statement. If  $\Gamma$  does not pass through  $p_4$ , then it is contained in the nonsingular locus of  $X'$ . Then, the argument in Step 2 of [5, Proof of Theorem 5.1.1] to our case, and as a result, we see that  $\Gamma$  is not a maximal center. This completes the proof.  $\square$

In the rest of this section, we exclude the remaining curves by applying the following results.

**Lemma 10.2.** *Let  $X$  be a  $\mathbb{Q}$ -Fano 3-fold with Picard number 1 and  $\Gamma \subset X$  an irreducible and reduced curve. Suppose that there is a movable linear system  $\mathcal{M}$  on  $X$  with the following properties.*

- (1)  $\Gamma$  is the unique base curve of  $\mathcal{M}$ .
- (2) A general member  $S \in \mathcal{M}$  is a normal surface.
- (3) For a general  $S \in \mathcal{M}$ ,  $(\Gamma^2)_S \leq 0$  and  $((-K_X)^2 \cdot S) - 2(-K_X \cdot \Gamma) + (\Gamma^2)_S \leq 0$ .

Then,  $\Gamma$  is not a maximal center.

*Proof.* Let  $\mathcal{H} \sim_{\mathbb{Q}} -nK_X$  be a movable linear system on  $X$ . It is enough to show that  $\text{mult}_{\Gamma} \mathcal{H} \leq 1$ . Let  $S \in \mathcal{M}$  be a general member so that it does not contain base curves of  $\mathcal{H}$  other than  $\Gamma$ . Then, we can write

$$(-K_X)|_S \sim_{\mathbb{Q}} \frac{1}{n} \mathcal{H}|_S = \frac{1}{n} \mathcal{L} + \gamma \Gamma,$$

where  $\mathcal{L}$  is a movable linear system on  $S$  and  $\gamma \geq \text{mult}_{\Gamma} \mathcal{H}$ . Since  $\mathcal{L}$  is nef, we have

$$0 \leq \frac{1}{n^2} (\mathcal{L}^2)_S = ((-K_X)|_S - \gamma \Gamma)_S^2 = ((-K_X)^2 \cdot S) - 2(-K_X \cdot \Gamma) \gamma + (\Gamma^2)_S \gamma^2.$$

The right-hand side of the above equation is a strictly decreasing function of  $\gamma$  for  $\gamma \geq 0$  since  $(-K_X \cdot \Gamma) > 0$  and  $(\Gamma^2)_S \leq 0$ . Thus, the inequality  $((-K_X)^2 \cdot S) - 2(-K_X \cdot \Gamma) + (\Gamma^2)_S \leq 0$  implies  $\gamma \leq 1$ . Therefore,  $\text{mult}_{\Gamma} \mathcal{H} \leq \gamma \leq 1$  and  $\Gamma$  is not a maximal center.  $\square$

**Lemma 10.3** ([13, Lemma 2.10]). *Let  $X$  be a  $\mathbb{Q}$ -Fano 3-fold with Picard number 1 and  $\Gamma \subset X$  an irreducible and reduced curve. Suppose that there is an effective divisor  $T$  on  $X$  containing  $\Gamma$  and a movable linear system  $\mathcal{M}$  on  $X$  whose base locus contains  $\Gamma$  with the following properties.*

- (1)  $T \sim_{\mathbb{Q}} mA$  for some rational number  $m \geq 1$ .
- (2) For a general member  $S \in \mathcal{M}$ ,  $S$  is a normal surface, the intersection  $S \cap T$  is contained in the base locus of  $\mathcal{M}$  set-theoretically and  $S \cap T$  is reduced along  $\Gamma$ .
- (3) Let  $S \in \mathcal{M}$  be a general member and let  $\Gamma_1, \dots, \Gamma_l$  be the irreducible and reduced curves contained in the base locus of  $\mathcal{M}$ . For each  $i = 1, \dots, l$ , there

is an effective 1-cycle  $\Delta_i$  on  $S$  such that  $(\Gamma \cdot \Delta_i)_S \geq (A \cdot \Delta_i)_S > 0$  and  $(\Gamma_j \cdot \Delta_i)_S \geq 0$  for  $j \neq i$ .

**10.2. Computation of intersection numbers of on a surface.** As we explained in the previous subsection, the exclusion of curves is reduced to the computation of intersection numbers on a surface. In this paper, the pullback of a Weil divisor and the intersection number of two Weil divisors on a normal surface are those in the sense of Mumford (see [11]). We explain ideas of the proof.

Let  $\Gamma \subset X'$  be an irreducible and reduced curve. We try to find a divisor  $T$  on  $X'$  and a linear system  $\mathcal{M}$  such that the assumptions of Lemma 10.3 hold. The crucial part here is to conclude  $(\Gamma \cdot \Delta_i) \geq \deg \Delta_i$ . Here, we use notation of Lemma 10.3. In most of the cases, we set  $\Delta_i = \Gamma_i$  and prove the inequality  $(\Gamma \cdot \Gamma_i) \geq \deg \Gamma_i$ . In many cases, the computation of  $(\Gamma \cdot \Gamma_i)$  will be done by counting the number of intersection points  $\Gamma \cap \Gamma_i$  along the nonsingular locus of a surface  $S \in \mathcal{M}$ . To this end, we need to know nonsingular locus of  $S$ .

**Lemma 10.4.** *Let  $V$  be a weighted hypersurface in  $\mathbb{P}(a_0, \dots, a_4)$  with defining polynomial  $F$  and  $\Gamma = (x_0 + f_0 = x_1 + f_1 = x_2 + f_2 = 0)$  for some  $f_i \in \mathbb{C}[x_0, \dots, x_4]$ . Suppose that  $a_0 \leq a_1$ . Let  $\Lambda \subset |\mathcal{O}_V(a_1)|$  be the linear system generated by  $x_1 + f_1$  and  $\{(x_0 + f_0) \prod x_i^{m_i} \mid m_i \geq 0, m_0 + \dots + m_4 = a_1 - a_0\}$ , and let  $S \in \Lambda$  be a general member. If  $V$  does not contain  $(x_0 + f_0 = x_1 + f_1 = 0)$  and  $(x_0 + f_0 = x_1 + f_1 = 0)_V$  is reduced along  $\Gamma$ , then  $S$  is quasismooth at any point of*

$$\Gamma \setminus \text{NQsm}(V) \cup \left( \text{Bs} |\mathcal{O}_{\mathbb{P}}(a_1 - a_0)| \cap \left( \frac{\partial F}{\partial x_0} = 0 \right) \right),$$

where  $\text{NQsm}(V)$  denotes the non-quasismooth locus of  $V$ .

*Proof.* After replacing  $x_0, x_1, x_2$ , we may assume  $f_0 = f_1 = f_2 = 0$ . Since  $\Gamma \subset V$ , we have  $F = x_0 g_0 + x_1 g_1 + x_2 g_2$  for some  $g_i \in \mathbb{C}[x_0, \dots, x_4]$ . Note that  $S$  is cut out on  $V$  by the section  $x_0 q + \lambda x_1$ , where  $q$  is a general homogeneous polynomial of degree  $a_1 - a_0$  and  $\lambda \in \mathbb{C}$  is general. We have

$$J_S|_{\Gamma} = \begin{pmatrix} g_0 & g_1 & g_2 & 0 & 0 \\ q & \lambda & 0 & 0 & 0 \end{pmatrix}.$$

We see that  $g_2$  is a non-zero polynomial and it does not vanish along  $\Gamma$  since  $(x_0 = x_1 = 0) \not\subset V$  and  $(x_0 = x_1 = 0)_V$  is reduced along  $\Gamma$ . It follows that  $\text{NQsm}(S)$  is contained in the finite set  $\Gamma \cap (g_2 = 0)$ . Let  $\mathbf{p} \in \text{NQsm}(S) \setminus \text{NQsm}(V)$ . Note that either  $g_0(\mathbf{p}) \neq 0$  or  $g_1(\mathbf{p}) \neq 0$  since  $V$  is quasismooth at  $\mathbf{p}$  and  $g_2(\mathbf{p}) = 0$ . If  $\mathbf{p} \notin \text{Bs} |\mathcal{O}_{\mathbb{P}}(a_1 - a_0)|$ , then  $q(\mathbf{p}) \neq 0$  for a general  $q$  and hence  $\text{rank } J_S(\mathbf{p}) = 2$  for a general  $\lambda$ . If  $(\partial F / \partial x_0)(\mathbf{p}) = g_0(\mathbf{p}) \neq 0$ , then  $\text{rank } J_S(\mathbf{p}) = 2$  for a general  $\lambda$  since  $g_1(\mathbf{p}) \neq 0$ . This completes the proof.  $\square$

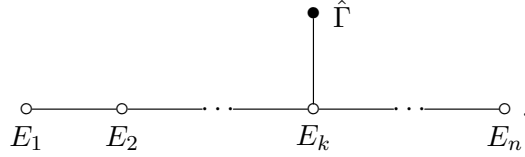
We encounter with the case where  $\Gamma \cap \Gamma_i = \{\mathbf{p}_4\}$ , which makes the computation of  $(\Gamma \cdot \Gamma_i)$  difficult. When we apply Lemma 10.2, we need to obtain the inequality in (3). In other words, we need to bound  $(\Gamma^2)_S$  from above. The computation of  $(\Gamma^2)_S$  is not so straightforward since  $\Gamma$  passes through the  $cA/n$  point  $\mathbf{p}_4$ . In these cases, the computation will be done by the following method.

**Definition 10.5.** Let  $(S, \mathbf{p})$  be a germ of a normal surface and  $\Gamma$  an irreducible and reduced curve on  $S$ . Let  $\hat{S} \rightarrow S$  be the minimal resolution of  $(S, \mathbf{p})$  and denote by  $E_1, \dots, E_m$  the prime exceptional divisors. We define  $G(S, \mathbf{p}, \Gamma)$  to be the dual

graph of  $E_1, \dots, E_m$  and the proper transform  $\hat{\Gamma}$  of  $\Gamma$  on  $\hat{S}$ : vertices of  $G(S, \mathbf{p}, \Gamma)$  corresponds to  $E_1, \dots, E_m$  and  $\hat{\Gamma}$ , and two vertices corresponding to  $E_i$  and  $E_j$  (resp.  $E_i$  and  $\hat{\Gamma}$ ) are joined by  $(E_i \cdot E_j)$ -ple edge (resp.  $(E_i \cdot \hat{\Gamma})$ -ple edge). We call  $G(S, \mathbf{p}, \Gamma)$  the *extended dual graph* of  $(S, \mathbf{p}, \Gamma)$ .

In this paper, we only treat germs  $(S, \mathbf{p})$  of singularity of type  $A_n$ .

**Definition 10.6.** We say that  $G(S, \mathbf{p}, \Gamma)$  is of type  $A_{n,k}$  if it is of the form



Here,  $\circ$  means that the corresponding exceptional divisor is a  $(-2)$ -curve. In other words,  $G(S, \mathbf{p}, \Gamma)$  is of type  $A_{n,k}$  if  $(S, \mathbf{p})$  is of type  $A_n$ ,  $(\hat{\Gamma} \cdot E_i) = 0$  for  $i \neq k$  and  $(\hat{\Gamma} \cdot E_k) = 1$ .

**Lemma 10.7.** Let  $S$  be a normal projective surface and  $\Gamma$  a nonsingular rational curve on  $S$ . Let  $\mathbf{p}$  be a singular point of  $S$  and suppose that  $S$  is nonsingular along  $\Gamma \setminus \{\mathbf{p}\}$ . If  $G(S, \mathbf{p}, \Gamma)$  is of type  $A_{n,k}$ , then

$$(\Gamma^2)_S = -2 - (K_S \cdot \Gamma)_S + \frac{k(n-k+1)}{n+1}.$$

*Proof.* Let  $\psi: \hat{S} \rightarrow S$  be the minimal resolution of  $S$  with prime exceptional divisors  $E_1, \dots, E_n$ . We denote by  $\hat{\Gamma}$  the proper transform of  $\Gamma$  on  $\hat{S}$ . We have  $\psi^*\Gamma = \hat{\Gamma} + a_1 E_1 + \dots + a_n E_n$  for some rational numbers  $a_1, \dots, a_n$ .

Suppose that  $G(S, \mathbf{p}, \Gamma)$  is of type  $A_{n,k}$ . We have

$$\begin{aligned} 0 &= (\varphi^*\Gamma \cdot E_i) = (\hat{\Gamma} \cdot E_i) + a_1(E_1 \cdot E_i) + \dots + a_n(E_n \cdot E_i) \\ &= \begin{cases} a_{i-1} - 2a_i + a_{i+1}, & \text{if } i \neq k, \\ 1 + a_{k-1} - 2a_k + a_{k+1}, & \text{if } i = k \end{cases} \end{aligned}$$

Here, we define  $a_0 = a_{n+1} = 0$ . By  $a_{i-1} - 2a_i + a_{i+1} = 0$  for  $i \neq k$ , we have  $a_i = ia_1$  for  $i = 1, 2, \dots, k$  and  $a_{n-i+1} = ia_n$  for  $i = 1, 2, \dots, n-k+1$ . In particular,  $a_k = ka_1 = (n-k+1)a_n$ , and thus  $a_n = \frac{k}{n-k+1}a_1$ . Now, by the equations

$$\begin{aligned} 0 &= (\varphi^*\Gamma \cdot E_i) = 1 + a_{k-1} - 2a_k + a_{k+1} \\ &= 1 + (k-1)a_1 - 2ka_1 + \frac{k(n-k)}{n-k+1}a_1 = 1 - \frac{n+1}{n-k+1}a_1, \end{aligned}$$

we have

$$a_1 = \frac{n-k+1}{n+1}, \quad a_k = ka_1 = \frac{k(n-k+1)}{n+1}.$$

The assertion follows from the combination of equations

$$(\Gamma^2) = (\hat{\Gamma} \cdot \psi^*\Gamma) = (\hat{\Gamma}^2) + a_k$$

and

$$(\hat{\Gamma}^2) = -2 + (K_{\hat{S}} \cdot \hat{\Gamma}) = -2 + (K_S \cdot \Gamma).$$

This completes the proof.  $\square$

In what follows, we sometimes consider a suitable weighted blowup  $\varphi: Y' \rightarrow X'$ , which is a divisorial extraction. In this case, we always denote by  $E$  its exceptional divisor and by  $\tilde{\Delta}$  the proper transform on  $Y'$  of a curve or a divisor  $\Delta \subset X'$ .

Finally, we define suitable coordinate change.

**Definition 10.8.** Let  $X' \subset \mathbb{P}(1, n, a_2, a_3, n)$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^*$  defined by a standard defining polynomial  $F' = w^2x_2x_3 + wf + g$ . An *admissible coordinate change* of  $X'$  is a coordinates change that defines an automorphism  $\theta$  of  $\mathbb{P}(1, n, a_2, a_3, n)$  such that  $\theta^*F' = \alpha(w^2x_2x_3 + wf' + g')$  for some non-zero  $\alpha \in \mathbb{C}$  and  $f', g' \in \mathbb{C}[x_0, \dots, x_3]$ .

A change of coordinates  $\theta$  is admissible if and only if  $\theta^*w = \beta w + (\text{other terms})$  for some non-zero  $\beta \in \mathbb{C}$ ,  $\theta^*x_j$  does not involve  $w$  for  $j = 0, 1, 2, 3$  and  $\theta^*(x_2x_3) = \gamma x_2x_3$  for some non-zero  $\gamma \in \mathbb{C}$ .

**10.3. Curves of degree 1 on  $X' \in \mathcal{G}'_6$ .** Let  $X' = X'_5 \subset \mathbb{P}(1, 1, 1, 2, 1)$  be a member of  $\mathcal{G}'_6$  with defining polynomial  $F' = w^2x_0y + wf_4 + g_5$  and  $\Gamma$  an irreducible and reduced curve of degree 1 on  $X'$  that passes through  $p_4$  but does not pass through the other singular points.

We claim that  $\Gamma$  is a WCI curve of type  $(1, 1, 2)$ . Indeed, the projection  $\pi: X' \dashrightarrow \mathbb{P}^3$  to the coordinates  $x_0, x_1, x_2, w$  induces the finite morphism  $\pi|_{\Gamma}: \Gamma \rightarrow \pi(\Gamma)$ . We have  $1 = \deg \Gamma = \deg(\pi|_{\Gamma}) \deg \pi(\Gamma)$ . Hence  $\pi|_{\Gamma}$  is an isomorphism and  $\pi(\Gamma)$  is a line in  $\mathbb{P}^3$ . It follows that there are linear forms  $\ell_1, \ell_2 \in \mathbb{C}[x_0, x_1, x_2, w]$  such that  $\Gamma \subset (\ell_1 = \ell_2 = 0)_{X'}$ . From this we see that  $\Gamma = (\ell_1 = \ell_2 = q = 0)$  for some  $q \in \mathbb{C}[x_0, x_1, x_2, y, z, w]$  of degree 2.

Let  $S$  and  $T$  be general members of the pencil  $|\mathcal{I}_{\Gamma}(A)|$ .

**Lemma 10.9.** *After an admissible coordinate change, we have  $T|_S = \Gamma + \Delta$ , where the pair  $(\Gamma, \Delta)$  of curves on  $S$  is one of the following.*

- (1)  $\Gamma = (x_1 = x_2 = y + wx_0 + \beta x_0^2 = 0)$  and  $\Delta = (x_1 = x_2 = wy + \nu x_0^3 = 0)$  for some  $\beta, \nu \in \mathbb{C}$ .
- (2)  $\Gamma = (x_1 = x_2 = y = 0)$  and  $\Delta = (x_1 = x_2 = w^2x_0 + wy + \lambda wx_0^2 + \nu x_0^3 = 0)$  for some  $\lambda, \mu \in \mathbb{C}$ .
- (3)  $\Gamma = (x_0 = x_1 = y + \beta x_2^2 = 0)$  and  $\Delta = (x_0 = x_1 = wy + \lambda wx_2^2 + \nu x_2^3 = 0)$  for some  $\beta, \lambda, \nu \in \mathbb{C}$  with  $\nu \neq 0$  and  $(\beta, \lambda) \neq (0, 0)$ .

*Proof.* Since  $y^2 \in f_4$ , we may assume that the coefficient of  $y^2$  in  $f_4$  is 1 and that there is no monomial divisible by  $y^2$  in  $g_5$  after replacing  $w$ .

Suppose  $\Gamma \not\subset (x_0 = 0)$ . Then, after replacing  $x_1, x_2$ , we may assume  $\Gamma = (x_1 = x_2 = y + \alpha wx_0 + \beta x_0^2 = 0)$  for some  $\alpha, \beta \in \mathbb{C}$ . We have

$$\bar{F}' := F'(x_0, 0, 0, y, w) = w^2x_0y + w(y^2 + \gamma yx_0^2 + \delta x_0^4) + \varepsilon yx_0^3 + \zeta x_0^5,$$

where  $\gamma, \dots, \zeta \in \mathbb{C}$ . Since  $\Gamma \subset X'$ , we have

$$\bar{F}' = (y + \alpha wx_0 + \beta x_0^2)(\eta w^2x_0 + \theta wy + \lambda wx_0^2 + \mu yx_0 + \nu x_0^3)$$

for some  $\eta, \dots, \nu \in \mathbb{C}$ . By comparing the coefficients of  $w^3x_0^2, w^2x_0y, w^2x_0^3, wy^2$  and  $y^2x_0$ , we have

$$\alpha\eta = 0, \eta + \alpha\theta = 1, \alpha\lambda + \beta\eta = 0, \theta = 1, \mu = 0.$$

If  $\alpha \neq 0$ , then  $\eta = 0, \alpha = \theta = 1, \lambda = 0, \mu = 0$ , and this case corresponds to (1). If  $\alpha = 0$ , then  $\eta = 1, \beta = 0, \theta = 1, \mu = 0$ , and this case corresponds to (2).

Suppose  $\Gamma \subset (x_0 = 0)$ . Then, after replacing  $x_1, x_2$ , we may assume  $\Gamma = (x_0 = x_1 = y + \alpha wx_2 + \beta x_2^2 = 0)$ . We have

$$\bar{F}' := F'(0, 0, x_2, y, w) = w(y^2 + \gamma yx_2^2 + \delta x_2^4) + \varepsilon yx_2^3 + \zeta x_2^5,$$

where  $\gamma, \dots, \zeta \in \mathbb{C}$ . Since  $\Gamma \subset X'$ , we have

$$\bar{F}' = (y + \alpha wx_2 + \beta x_2^2)(\theta wy + \lambda wx_2^2 + \mu yx_2 + \nu x_2^3)$$

for some  $\theta, \dots, \nu \in \mathbb{C}$ . By comparing the coefficients of  $w^2x_2y$ ,  $w^2x_2^3$ ,  $wy^2$  and  $y^2x_2$ , we have

$$\alpha\theta = 0, \alpha\lambda = 0, \theta = 1, \mu = 0,$$

and hence  $\alpha = \mu = 0$ ,  $\theta = 1$ . We claim that  $\nu \neq 0$  and  $(\beta, \lambda) \neq (0, 0)$ . Suppose  $\nu = 0$ . Then,  $\varepsilon = \nu = 0$  and  $\zeta = \beta\nu = 0$ . Since  $\varepsilon$  and  $\zeta$  are the coefficients of  $x_2^3$  and  $x_2^5$  in  $b_3$  and  $b_5$ , respectively, this implies that  $x_0 = b_3 = b_5 = 0$  has a solution  $(x_0, x_1, x_2) = (0, 0, 1)$ . This is impossible by Condition 2.12. Thus, this case corresponds to (3). Suppose  $(\beta, \lambda) = (0, 0)$ . Then  $\gamma = \beta + \lambda = 0$  and  $\delta = \beta\lambda = 0$ , which implies that  $x_0 = a_2 = a_4 = 0$  has a solution  $(x_0, x_1, x_2) = (0, 0, 1)$ . This is again impossible by Condition 2.12.  $\square$

Note that  $S$  is nonsingular along  $\Gamma \setminus \{\mathfrak{p}_4\}$  by Lemma 10.4.

**Lemma 10.10.** *Let  $T|_S = \Gamma + \Delta$  be as in Lemma 10.9. Then the following assertions hold.*

- (1) *If  $\Delta$  is irreducible, then  $(\Gamma \cdot \Delta)_S \geq \deg \Delta$ .*
- (2) *If  $\Delta$  is reducible, then it splits as  $\Delta = \Delta_1 + \Delta_2$ , where  $\Delta_1, \Delta_2$  are irreducible and reduced curves of degree respectively 1 and  $1/2$ , respectively, such that  $(\Gamma \cdot \Delta_i)_S \geq \deg \Delta_i$  for  $i = 1, 2$ .*

*In particular,  $\Gamma$  is not a maximal center.*

*Proof.* Suppose that we are in case (1) of Lemma 10.9. We see that  $\Delta$  is clearly reduced and it is irreducible if and only if  $\nu \neq 0$ . If  $\Delta$  is irreducible, then it intersects  $\Gamma$  at two nonsingular points so that  $(\Gamma \cdot \Delta)_S \geq 2 > \deg \Delta$ . If  $\Delta$  is reducible, then  $\Delta = \Delta_1 + \Delta_2$ , where  $\Delta_1 = (x_1 = x_2 = y = 0)$  and  $\Delta_2 = (x_1 = x_2 = w = 0)$ . Both  $\Delta_1$  and  $\Delta_2$  intersect  $\Gamma$  at a nonsingular point so that  $(\Gamma \cdot \Delta_i)_S \geq 1 \geq \deg \Delta_i$  for  $i = 1, 2$ . The computation of intersection numbers in case (2) of Lemma 10.9 can be done in the same way and we omit it.

Suppose that we are in case (3) of Lemma 10.9. Then  $\Delta$  is irreducible and reduced since  $\nu \neq 0$ . If  $\lambda \neq \beta$ , then  $\Delta$  intersects  $\Gamma$  at two points away from  $\mathfrak{p}_4$ , so that  $(\Gamma \cdot \Delta)_S \geq 2 > \deg \Delta$ . In the following, we assume  $\lambda = \beta$ . Note that  $\beta \neq 0$  in this case since  $(\beta, \lambda) \neq (0, 0)$ . In this case,  $\mathfrak{p}_4$  is the unique intersection point of  $\Gamma$  and  $\Delta$  and we cannot compute  $(\Gamma \cdot \Delta)_S$  directly. We will compute  $(\Gamma^2)_S$  by determining the extended dual graph  $G(S, \mathfrak{p}_4, \Gamma)$ .

Let  $\varphi: Y' \rightarrow X'$  be the weighted blowup of  $X'$  at  $\mathfrak{p}_4$  with weight  $\text{wt}(x_0, x_1, x_2, y) = (2, 1, 1, 2)$  and with exceptional divisor  $E$ . We define  $\psi := \varphi|_{\tilde{S}}: \tilde{S} \rightarrow S$  and set  $E_\psi = E|_{\tilde{S}}$ . We see that  $S$  is cut out on  $X'$  by the section

$$s := x_0\ell_0 + x_1\ell_1 + \mu(y + \beta x_2^2),$$

where  $\ell_1, \ell_2$  are general linear forms in  $x_0, x_1, x_2, w$  and  $\mu \in \mathbb{C}$  is general. Note that  $K_{\tilde{S}} = \psi^*K_S$  by adjunction since  $K_{Y'} = \varphi^*K_{X'} + E$  and  $s$  vanishes along  $E$  to order 1. Let  $\xi_i \in \mathbb{C}$  be the coefficient of  $w$  in  $\ell_i$ . Let  $G$  and  $H$  be the  $\varphi$ -weight = 4 and

5 parts of  $F(x_0, x_1, x_2, y, 1) = x_0y + f_4 + g_5$ , respectively. We have  $G = x_0y + \bar{f}_4$ . Here,  $\bar{f}_4 = f_4(0, x_1, x_2, y)$ . Then, we have an isomorphism

$$E_\psi \cong (x_0y + \bar{f}_4 = x_1 = 0) \subset \mathbb{P}(2_{x_0}, 1_{x_1}, 1_{x_2}, 2_y),$$

and

$$J_\psi = \begin{pmatrix} y & \frac{\partial \bar{f}_4}{\partial x_1} & \frac{\partial \bar{f}_4}{\partial x_2} & \frac{\partial \bar{f}_4}{\partial y} & H \\ 0 & \xi_1 & 0 & 0 & \xi_0x_0 + x_1\bar{\ell}_1 + \mu(y + \beta x_2^2) \end{pmatrix}.$$

Set

$$\Sigma := \left( y = \frac{\partial \bar{f}_4}{\partial x_2} = \frac{\partial \bar{f}_4}{\partial y} = 0 \right) \cap E_\psi = \left( x_1 = y = \bar{f}_4 = \frac{\partial \bar{f}_4}{\partial x_2} = \frac{\partial \bar{f}_4}{\partial y} = 0 \right).$$

We see that  $\Sigma = \emptyset$  if and only if  $x_2^4 \in f_4$ . The latter holds true since

$$F(0, 0, x_2, y, w) = (y + \beta x_2^2)(wy + \beta wx_2^2 + \nu x_2^3)$$

and  $\beta \neq 0$ . This shows that  $J_\psi$  is of rank 2 at every point of  $E_\psi$  and hence singular points of  $\tilde{S}$  consists of two points  $\mathbf{q}_1 = (1:0:0:0)$  and  $\mathbf{q}_2 = (1:0:0:-1)$  both of type  $A_1$ . Note that  $\tilde{\Gamma}$  intersects  $E_\psi$  at  $(0:0:1:-\beta) \neq \mathbf{q}_1, \mathbf{q}_2$ . By considering the blowups of  $\tilde{S}$  at  $\mathbf{q}_1$  and  $\mathbf{q}_2$ , we see that  $G(S, \mathbf{p}_4, \Gamma)$  is of type  $A_{3,2}$ . Thus, by Lemma 10.7,

$$(\Gamma^2)_S = -2 - \deg \Gamma + \frac{3}{2} = -\frac{3}{2}.$$

By taking the intersection number of  $\Gamma$  and  $T|_S = \Gamma + \Delta$ , we have

$$1 = (\Gamma \cdot T|_S)_S = (\Gamma^2)_S + (\Gamma \cdot \Delta)_S = -\frac{3}{2} + (\Gamma \cdot \Delta)_S,$$

and thus  $(\Gamma \cdot \Delta)_S = \frac{5}{2} > \deg \Delta$ . This completes the proof.  $\square$

**10.4. Curves of degree 2 on  $X' \in \mathcal{G}'_6$ .** Let  $X' = X'_5 \subset \mathbb{P}(1, 1, 1, 2, 1)$  be a member of  $\mathcal{G}'_6$  and  $\Gamma \subset X'$  an irreducible and reduced curve of degree 2 that passes through  $\mathbf{p}_4$  but does not pass through the other singular points. We see that  $\Gamma$  is a WCI curve of type either  $(1, 1, 4)$  or  $(1, 2, 2)$ .

**Lemma 10.11.** *No curve of type  $(1, 1, 4)$  is a maximal center.*

*Proof.* Let  $\Gamma \subset X'$  be a curve of type  $(1, 1, 4)$ . We claim that  $\Gamma \not\subset (x_0 = 0)$ . Indeed, if  $\Gamma \subset (x_0 = 0)$ , then we may assume  $\Gamma = (x_0 = x_1 = h_4 = 0)$  for some  $h_4 \in \mathbb{C}[x_2, y, w]$  after replacing  $x_0, x_1$ . We have

$$G := F(0, 0, x_2, y, w) = w(y^2 + \alpha yx_2^2 + \beta x_2^4) + \gamma yx_2^3 + \delta x_2^5$$

for some  $\alpha, \dots, \delta \in \mathbb{C}$ . Since  $\Gamma \subset X'$ ,  $G$  is reducible, which is equivalent to the condition  $\gamma = \delta = 0$ . Then, we have  $h_4 = y^2 + \alpha yx_2^2 + \beta x_2^4$ , which implies that  $\Gamma$  is either reducible or non-reduced. This is a contradiction.

Thus,  $\Gamma \not\subset (x_0 = 0)$  and we may assume  $\Gamma = (x_1 = x_2 = h_4 = 0)$  for some  $h_4 \in \mathbb{C}[x_0, y, w]$  after replacing  $x_0, x_1$ . Let  $S$  and  $T$  be general members of the pencil  $|\mathcal{I}_\Gamma(A)|$  generated by  $x_1, x_2$ . By an explicit computation, we have  $h_4 = y^2 + ywx_0 + \alpha yx_0^2 + \beta x_0^4$  for some  $\alpha, \beta \in \mathbb{C}$  and  $T|_S = \Gamma + \Delta$ , where  $\Delta = (x_1 = x_2 = w + \gamma x_0 = 0)$  for some  $\gamma \in \mathbb{C}$ . Note that  $S$  is nonsingular along  $\Gamma \setminus \{\mathbf{p}_4\}$  by Lemma 10.4. We see that  $\Gamma$  intersects  $\Delta$  at two nonsingular points so that  $(\Gamma \cdot \Delta) \geq 2 > (A \cdot \Delta) = 1/2$ . Therefore,  $\Gamma$  is not a maximal center.  $\square$

In the following, we treat the case where  $\Gamma$  is of type  $(1, 2, 2)$ . We write the defining polynomial of  $X'$  as  $F' = w^2x_0y + w(y^2 + ya_2 + a_4) + yb_3 + b_5$ , where  $a_i, b_i \in \mathbb{C}[x_0, x_1, x_2]$ . For a polynomial  $h = h(x_0, x_1, x_2)$ , we set  $\bar{h} = h(0, x_1, x_2)$  and  $\tilde{h} = h(x_0, x_1, 0)$ .

**Lemma 10.12.** *Let  $\Gamma \subset X'$  be a curve of type  $(1, 2, 2)$  passing through  $\mathfrak{p}_4$ . Then, by an admissible change of coordinates,  $\Gamma$  is one of the following.*

- (1)  $(x_0 = y - c_2 = wx_1 + d_2 = 0)$  for some  $c_2, d_2 \in \mathbb{C}[x_1, x_2]$  such that  $x_1 \nmid d_2$ .
- (2)  $(x_2 = y - c_2 = wx_0 + d_2 = 0)$  for some  $c_2, d_2 \in \mathbb{C}[x_0, x_1]$  such that  $x_0 \nmid d_2$ . In this case, if  $a_4(0, 1, 0) = 0$ , then either  $c_2(1, 0) \neq 0$  or  $a_2(0, 1, 0) \neq d_2(1, 0)$ .
- (3)  $(x_2 = y - \lambda wx_0 - c_2 = wx_1 + d_2 = 0)$  for some  $\lambda \in \mathbb{C}$ ,  $c_2, d_2 \in \mathbb{C}[x_0, x_1]$  such that  $x_1 \nmid d_2$ .

*Proof.* Suppose  $\Gamma \subset (x_0 = 0)$ . We will show that we are in case (1). We can write  $\Gamma = (x_0 = y - wc_1 - c_2 = wd_1 + d_2 = 0)$  for some  $c_1, c_2, d_1, d_2 \in \mathbb{C}[x_1, x_2]$ . Since  $\Gamma$  is irreducible and reduced, we have  $d_1 \neq 0$  and  $d_2$  is not divisible by  $d_1$ . Hence, we may assume  $d_1 = x_1$  and  $x_1 \nmid d_2$ . After replacing  $y - wc_1 - c_2$  with  $(y - wc_1 - c_2) - \eta(wx_1 + d_2)$  for some  $\eta \in \mathbb{C}$ , we may assume that either  $c_1 = 0$  or  $x_1 \nmid c_1$ . Since  $\Gamma \subset X'$ , there exists  $h = h(x_1, x_2, w)$  such that  $F'(0, x_1, x_2, wc_1 + c_2, w) = (wx_1 + d_2)h$ . We have

$$F'(0, x_1, x_2, wc_1 + c_2, w) = w^3c_1^2 + \dots,$$

where  $\dots$  consists of low degree terms with respect to  $w$ . By writing  $h = w^2e_1 + we_2 + e_3$  for some  $e_i \in \mathbb{C}[x_1, x_2]$  and comparing terms divisible by  $w^3$ , we have  $w^3c_1^2 = w^3x_1e_1$ . This shows that  $c_1$  is divisible by  $x_1$  and thus  $c_1 = 0$ . Thus we are in case (1).

In the following, we assume  $\Gamma \not\subset (x_0 = 0)$ . Then, we may assume  $\Gamma \subset (x_2 = 0)$  and  $\Gamma = (x_2 = y - wc_1 - c_2 = wd_1 + d_2 = 0)$  for some  $c_1, c_2, d_1, d_2 \in \mathbb{C}[x_0, x_1]$ . As in the above argument, we see that  $d_1 \neq 0$ ,  $d_1 \nmid d_2$  and we may assume that either  $c_1 = 0$  or  $d_1 \nmid c_1$ . If  $d_1$  is not proportional to  $x_0$ , then we may assume  $d_1 = x_1$  and thus we are in case (3). Suppose that  $d_1$  is proportional to  $x_0$ . Then we may assume  $d_1 = x_0$ . Since  $\Gamma \subset X'$ , there exists  $h = h(x_0, x_1, w)$  such that  $F'(x_0, x_1, 0, wc_1 + c_2, w) = (wx_0 + d_2)h$ . We have

$$\begin{aligned} F'(x_0, x_1, 0, wc_1 + c_2, w) &= w^3c_1(x_0 + c_1) + w^2(2c_1c_2 + c_1\tilde{a}_2 + x_0c_2) \\ &\quad + w(c_2^2 + \tilde{a}_2c_2 + \tilde{a}_4 + c_1\tilde{b}_3) + \tilde{b}_3c_2 + \tilde{b}_5. \end{aligned}$$

By writing  $h = w^2e_1 + we_2 + e_3$ , where  $e_i \in \mathbb{C}[x_0, x_1]$ , and comparing terms divisible by  $w^3$ , we have  $c_1(x_0 + c_1) = x_0e_1$ . If  $c_1 \neq 0$ , then  $e_1 \mid c_1$  since  $x_0 \nmid c_1$  by our choice of  $c_1$ . But then the equation  $c_1(x_0 + c_1) = x_0e_1$  implies  $x_0 \mid c_1$ . This is a contradiction and we have  $c_1 = e_1 = 0$ . Thus we are in case (2). It remains to show that if  $a_4(0, 1, 0) = 0$ , then either  $c_2(1, 0) \neq 0$  or  $a_2(0, 1, 0) \neq d_2(1, 0)$ . Since  $c_1 = e_1 = 0$ , we have  $e_2 = c_2$  and the equations

$$(11) \quad c_2^2 + \tilde{a}_2c_2 + \tilde{a}_4 = x_0e_3 + d_2c_2 \text{ and } \tilde{b}_3c_2 + \tilde{b}_5 = d_2e_3.$$

Let  $\alpha_2, \alpha_3$  and  $\beta_5$  be the coefficients of  $x_1^2, x_0x_1^3$  and  $x_1^5$  in  $\tilde{a}_2, \tilde{a}_4$  and  $\tilde{b}_5$ , respectively. Now we assume  $a_4(0, 1, 0) = c_2(1, 0) = a_2(0, 1, 0) - d_2(1, 0) = 0$ . This implies that the coefficients of  $x_1^2$  in  $d_2$  is  $\alpha_2$ . Let  $\varepsilon$  be the coefficient of  $x_1^3$  in  $e_3$ . By comparing the coefficients of  $x_0x_1^3$  in the first equation of (11), we have  $\alpha_2\gamma + \alpha_2 = \varepsilon + \alpha_2\gamma$  since  $x_1 \mid c_2$ . By comparing the coefficients of  $x_1^5$  in the second equation in (11), we have  $\beta_5 = \alpha_2\varepsilon$ . This shows  $\beta_5 = \alpha_2\alpha_3$  and thus  $(x_0, x_1, x_2) = (0, 1, 0)$  is a solution of



$x_0 = a_4 = b_5 - a_2 \partial a_4 / \partial x_0 = 0$ . This is impossible by Condition 2.12 and the proof is completed.  $\square$

Let  $\varphi: Y' \rightarrow X'$  be the weighted blowup of  $X'$  at  $\mathfrak{p}_4$  with  $\text{wt}(x_0, x_1, x_2, y) = (2, 1, 1, 2)$ . Let  $S \in |\mathcal{I}_\Gamma(2A)|$  be a general member. We set  $\psi = \varphi|_{\tilde{S}}: \tilde{S} \rightarrow S$  and  $E_\psi = E|_{\tilde{S}}$ . Note that  $K_{\tilde{S}} = \psi^* K_S$ . Let  $G$  and  $H$  be the  $\varphi$ -weight  $= 4$  and  $= 5$  parts of  $F'(x_0, x_1, x_2, y, 1)$ , respectively. We write  $a_2 = \bar{a}_2 + x_0 a_1$  and  $a_4 = \bar{a}_4 + x_0 a_3$ , where  $a_1, a_3 \in \mathbb{C}[x_1, x_2]$ . Then, we have  $G = x_0 y + y^2 + y \bar{a}_2 + \bar{a}_4$  and  $H = y x_0 \bar{a}_1 + x_0 \bar{a}_3 + y \bar{b}_3 + \bar{b}_5$ . We see that  $E$  is isomorphic to  $(G = 0) \subset \mathbb{P}(2, 1, 1, 2)$ .

We compute  $(\Gamma^2)_S$  by determining the extended resolution graph of  $(S, \mathfrak{p}_4, \Gamma)$ .

**Lemma 10.13.** *The type of the extended resolution graph  $G(S, \mathfrak{p}_4, \Gamma)$  is one of  $A_{3,1}$ ,  $A_{3,2}$  and  $A_{4,2}$ . In particular,  $(\Gamma^2) \leq -14/5$  and  $\Gamma$  is not a maximal center.*

*Proof.* It is easy to compute  $(\Gamma^2)_S$  once  $G(S, \mathfrak{p}_4, \Gamma)$  is determined. Indeed, if  $G(S, \mathfrak{p}_4, \Gamma)$  is of type one of  $A_{3,1}$ ,  $A_{3,2}$  and  $A_{4,2}$ , then, by Lemma 10.7, we have

$$(\Gamma^2)_S \leq -2 - \deg \Gamma + \frac{6}{5} = -\frac{14}{5}.$$

Thus

$$(A^2 \cdot S) - 2(A \cdot \Gamma) + (\Gamma^2)_S \leq 5 - 4 - \frac{14}{5} < 0,$$

and, by Lemma 10.2,  $\Gamma$  is not a maximal center. The rest is devoted to the determination of  $G(S, \mathfrak{p}_4, \Gamma)$ .

Suppose that we are in case (1) of Lemma 10.12. Then, the surface  $S$  is cut out on  $X'$  by the section

$$s := x_0(\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \beta w) + \gamma(y - c_2) + \delta(wx_1 + d_2),$$

where  $\alpha_i, \beta, \gamma, \delta \in \mathbb{C}$  are general, and we have an isomorphism

$$E_\psi \cong (x_0 y + y^2 + y \bar{a}_2 + \bar{a}_4 = x_1 = 0) \subset \mathbb{P}(2_{x_0}, 1_{x_1}, 1_{x_2}, 2_y).$$

We have

$$J_\psi = \begin{pmatrix} \frac{\partial G}{\partial x_0} & \frac{\partial G}{\partial x_1} & \frac{\partial G}{\partial x_2} & \frac{\partial G}{\partial y} & H \\ 0 & \delta & 0 & 0 & \beta x_0 + \gamma(y - c_2) + \delta d_2 \end{pmatrix}.$$

It is clear that  $J_\psi$  is of rank 2 except possibly along

$$\Sigma := \left( \frac{\partial G}{\partial x_0} = \frac{\partial G}{\partial x_2} = \frac{\partial G}{\partial y} = 0 \right) \cap E_\psi = \left( x_1 = y = \bar{a}_4 = x_0 + \bar{a}_2 = \frac{\partial \bar{a}_4}{\partial x_1} = 0 \right).$$

Note that  $\Sigma \neq \emptyset$  if and only if  $x_1 \mid \bar{a}_4$ , and that  $\Sigma = \{\mathbf{q}_0\}$ , where  $\mathbf{q}_0 = (-\bar{a}_2(0, 1):0:1:0)$ , if  $\Sigma \neq \emptyset$ . We set  $\mathbf{q}_1 = (1:0:0:0)$  and  $\mathbf{q}_2 = (1:0:0:-1)$ . Clearly  $\mathbf{q}_1$  and  $\mathbf{q}_2$  are  $A_1$  points of  $\tilde{S}$ , and  $\tilde{\Gamma}$  intersects  $E_\psi$  at a single point  $(0:0:1:c_2(0, 1)) \in E_\psi$ .

Assume that  $E_\psi$  is irreducible. This is equivalent to the condition  $x_1 \nmid \bar{a}_4$ . Then  $\Sigma = \emptyset$  and  $\text{Sing}(\tilde{S}) = \{\mathbf{q}_1, \mathbf{q}_2\}$ . Moreover,  $\tilde{\Gamma}$  does not pass through  $\mathbf{q}_1, \mathbf{q}_2$ . Hence, by considering the blowup of  $\tilde{S}$  at  $\mathbf{q}_1$  and  $\mathbf{q}_2$ , we see that  $G(S, \mathfrak{p}_4, \Gamma)$  is of type  $A_{3,2}$ .

Assume that  $E_\psi$  is reducible, that is,  $x_1 \mid \bar{a}_4$ . We have  $E_\psi = E_1 + E_2$ , where  $E_1 = (x_1 = y = 0)$  and  $E_2 = (x_1 = x_0 + y + \bar{a}_2 = 0)$ . Note that  $\mathbf{q}_1 \in E_1$ ,  $\mathbf{q}_2 \in E_2$  and  $\mathbf{q}_0$  is the intersection point  $E_1 \cap E_2$ . Note also that  $x_1 \nmid \bar{a}_2$  because otherwise  $x_0 = a_2 = a_4 = 0$  has a solution  $(x_0, x_1, x_2) = (0, 0, 1)$ . This in particular implies that  $\tilde{\Gamma}$  does not pass through  $\mathbf{q}_0$ . We have

$$J_\psi(\mathbf{q}_0) = \begin{pmatrix} 0 & \frac{\partial G}{\partial x_0}(\mathbf{q}_0) & 0 & 0 & (-\bar{a}_2 \bar{a}_3 + \bar{b}_5)(\mathbf{q}_0) \\ 0 & \delta & 0 & 0 & (-\beta \bar{a}_2 - \gamma c_2 + \delta d_2)(\mathbf{q}_0) \end{pmatrix}.$$

If  $(-\bar{a}_2\bar{a}_3 + \bar{b}_5)(\mathbf{q}_0) = 0$ , then  $x_0 = a_4 = b_5 - a_3\partial a_4/\partial x_0 = 0$  has a solution  $(x_0, x_1, x_2) = (0, 0, 1)$ . This is impossible by Condition 2.12. Hence  $(-\bar{a}_2\bar{a}_3 + \bar{b}_5)(\mathbf{q}_0) \neq 0$  and this implies  $\text{rank } J_\psi(\mathbf{q}_0) = 2$  since  $\delta$  is general. It follows  $\text{Sing}(\tilde{S}) = \{\mathbf{q}_1, \mathbf{q}_2\}$ . By considering blowups of  $\tilde{S}$  at  $\mathbf{q}_1$  and  $\mathbf{q}_2$ , we see that  $G(S, \mathbf{p}_4, \Gamma)$  is of type  $A_{4,2}$ .

Suppose that we are in case (2) of Lemma 10.12. The surface  $S$  is cut out on  $X'$  by the section

$$s := x_2(\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \beta w) + \gamma(y - c_2) + \delta(wx_0 + d_2),$$

where  $\alpha_i, \beta, \gamma, \delta \in \mathbb{C}$  are general, and we have an isomorphism

$$E_\psi \cong (x_0 y + y^2 + y\bar{a}_2 + \bar{a}_4 = x_2 = 0) \subset \mathbb{P}(2_{x_0}, 1_{x_1}, 1_{x_2}, 2_y).$$

We have

$$J_\psi = \begin{pmatrix} \frac{\partial G}{\partial x_0} & \frac{\partial G}{\partial x_1} & \frac{\partial G}{\partial x_2} & \frac{\partial G}{\partial y} & H \\ 0 & 0 & \beta & 0 & x_2(\alpha_1 x_1 + \alpha_2 x_2) + \gamma(y - \bar{c}_2) + \delta(x_0 + \bar{d}_2) \end{pmatrix}.$$

It is clear that  $J_\psi$  is of rank 2 outside the set

$$\Sigma := \left( \frac{\partial G}{\partial x_0} = \frac{\partial G}{\partial x_1} = \frac{\partial G}{\partial y} = 0 \right) \cap E_\psi = \left( y = \frac{\partial \bar{a}_4}{\partial x_1} = x_0 + \bar{a}_2 = \bar{a}_4 = x_2 = 0 \right).$$

Note that  $\Sigma \neq \emptyset$  if and only if  $x_2 \mid \bar{a}_4$  and  $\Sigma = \{\mathbf{q}_0\}$ , where  $\mathbf{q}_0 = (-\bar{a}_2(1, 0) : 1 : 0 : 0) \in \mathbb{P}(2, 1, 1, 2)$ , if  $\Sigma \neq \emptyset$ . Clearly  $\mathbf{q}_1 := (1 : 0 : 0 : 0)$  and  $\mathbf{q}_2 := (1 : 0 : 0 : -1)$  are  $A_1$  points of  $\tilde{S}$ . Note also that  $\tilde{\Gamma}$  intersects  $E_\psi$  at a single point  $(-d_2(1, 0) : 1 : 0 : c_2(1, 0))$ .

Assume that  $E_\psi$  is irreducible, which is equivalent to the condition  $x_2 \nmid \bar{a}_4$ . Then  $\Sigma = \emptyset$  and  $\text{Sing}(\tilde{S}) = \{\mathbf{q}_1, \mathbf{q}_2\}$ . Hence  $G(S, \mathbf{p}_4, \Gamma)$  is of type  $A_{3,2}$ .

Assume that  $E_\psi$  is reducible, that is,  $x_2 \mid \bar{a}_4$ . Then  $\Sigma = \{\mathbf{q}\}$  and  $E_\psi = E_1 + E_2$ , where  $E_1 = (y = x_2 = 0)$  and  $E_2 = (x_0 + y + \bar{a}_2 = x_2 = 0)$ . Note that  $\mathbf{q}_1 \in E_1$ ,  $\mathbf{q}_2 \in E_2$  and  $E_1 \cap E_2 = \{\mathbf{q}_0\}$ . Note also that  $\tilde{\Gamma}$  does not pass through  $\mathbf{q}_0$  since either  $c_2(1, 0) \neq 0$  or  $\bar{a}_2(1, 0) = a_2(0, 1, 0) \neq d_2(1, 0)$  by Lemma 10.12. By the same argument as in case (1), we see that  $H(\mathbf{q}_0) = (-\bar{a}_2\bar{a}_3 + \bar{b}_5)(\mathbf{q}_0) \neq 0$ , and hence  $\text{rank } J_\psi(\mathbf{q}_0) = 2$  since  $\beta$  is general. It follows that  $\text{Sing}(\mathbf{p}_4) = \{\mathbf{q}_1, \mathbf{q}_2\}$  and  $G(S, \mathbf{p}_4, \Gamma)$  is of type  $A_{4,2}$ .

Suppose that we are in case (3) of Lemma 10.12. The surface  $S$  is cut out on  $X'$  by the section

$$s := x_2(\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \beta w) + \gamma(y - \lambda w x_0 - c_2) + \delta(wx_1 + d_2),$$

where  $\alpha_i, \beta, \gamma, \delta \in \mathbb{C}$  are general. We have an isomorphism

$$E_\psi \cong (x_0 y + y^2 + y\bar{a}_2 + \bar{a}_4 = \delta x_1 + \beta x_2 = 0) \subset \mathbb{P}(2_{x_0}, 1_{x_1}, 1_{x_2}, 2_y),$$

which is irreducible since  $\bar{a}_4 \neq 0$  and  $\beta, \delta$  are general. We have

$$J_\varphi = \begin{pmatrix} \frac{\partial G}{\partial x_0} & \frac{\partial G}{\partial x_1} & \frac{\partial G}{\partial x_2} & \frac{\partial G}{\partial y} & H \\ 0 & \delta & \beta & 0 & x_2(\alpha_1 x_1 + \alpha_2 x_2) + \gamma(y - \lambda x_0 - \bar{c}_2) + \delta \bar{d}_2 \end{pmatrix}.$$

We see that

$$\left( \frac{\partial G}{\partial x_0} = \frac{\partial G}{\partial y} = 0 \right) \cap E_\psi = (y = x_0 + \bar{a}_2 = \bar{a}_4 = \delta x_1 + \beta x_2 = 0) = \emptyset$$

for a general  $\beta, \delta$ . This shows that  $J_\varphi$  is of rank 2 at every point of  $E_\varphi$  and thus  $\tilde{S}$  has only two singular points  $\mathbf{q}_1 = (1 : 0 : 0 : 0)$  and  $\mathbf{q}_2 = (1 : 0 : 0 : -1)$  of type  $A_1$ . We

see that  $\tilde{\Gamma}$  intersects  $E_\varphi$  transversally at  $\mathbf{q}_1$ . Hence, by considering the blow-up of  $\tilde{S}$  at  $\mathbf{q}_1$  and  $\mathbf{q}_2$ , we see that  $G(S, \mathbf{p}, \Gamma)$  is of type  $A_{3,1}$ . This completes the proof.  $\square$

**10.5. Curves of degree  $1/2$  on  $X' \in \mathcal{G}'_7$ .** Let  $X' = X'_6 \subset \mathbb{P}(1, 1, 1, 2, 2)$  be a member of  $\mathcal{G}'_7$  with defining polynomial  $F' = w^2x_0x_1 + wf_4 + g_6$  and let  $\Gamma \subset X'$  be an irreducible and reduced curve of degree  $1/2$  that passes through  $\mathbf{p}_4$  but does not pass through the other singular points. We see that  $\Gamma$  is a WCI curve of type  $(1, 1, 2)$  and it is contained in either  $(x_0 = 0)$  or  $(x_1 = 0)$ . By Condition 2.12,  $F'$  can be written as  $F' = w^2x_0x_1 + w(y^2 + ya_2 + a_4) + yb_4 + b_5$  for some  $a_i, b_i \in \mathbb{C}[x_0, x_1, x_2]$ . Let  $S, T$  be a general member of  $|\mathcal{I}_\Gamma(A)|$ .

**Lemma 10.14.** *We have  $T|_S = \Gamma + \Delta$ , where  $\Delta$  is an irreducible and reduced curve of degree 1. Moreover, one of the following hold.*

- (1)  $\Delta$  intersects  $\Gamma$  away from  $\mathbf{p}_4$ .
- (2) After an admissible coordinates change,  $\Gamma = (x_0 = x_2 = y = 0)$  and  $\Delta = (x_0 = x_2 = wy + \xi x_0^4 = 0)$  for some nonzero  $\xi \in \mathbb{C}$ .

*Proof.* Suppose  $\Gamma \subset (x_0 = x_1 = 0)$ . Then,  $\Gamma = (x_0 = x_1 = y + \gamma x_2^2 = 0)$  for some  $\alpha \in \mathbb{C}$ . We have

$$\bar{F}' := F(0, 0, x_2, y, w) = w(y^2 + \alpha_2 y x_2^2 + \alpha_4 x_2^4) + \beta_4 y x_2^4 + \beta_6 x_2^6,$$

where  $\alpha_i$  and  $\beta_i$  are the coefficients of  $x_2^i$  in  $a_i$  and  $b_i$ , respectively. Since  $\Gamma \subset X'$ , we have

$$\bar{F}' = (y + \gamma x_2^2)(wy + \delta w x_2^2 + \varepsilon x_2^4)$$

for some  $\delta, \varepsilon \in \mathbb{C}$ . By comparing the coefficients of each term of  $\bar{F}'$ , we have  $\alpha_2 = \gamma + \delta$ ,  $\alpha_4 = \gamma\delta$ ,  $\beta_4 = \varepsilon$  and  $\beta_6 = \gamma\varepsilon$ . Note that  $\Delta = (x_0 = x_1 = wy + \delta w x_2^2 + \varepsilon x_2^4 = 0)$ , which is irreducible if and only if  $\varepsilon \neq 0$ . If  $\varepsilon = 0$ , then  $\beta_4 = \beta_6 = 0$ , which in particular implies that  $x_0 = b_4 = b_6 = 0$  has a solution  $(x_0, x_1, x_2) = (0, 0, 1)$ . This is impossible by Condition 2.12 and hence  $\varepsilon \neq 0$ . If  $\gamma = \delta$ , then  $4\alpha_4 - \alpha_2^2 = 0$ , which implies that  $x_0 = x_1 = 4a_4 - a_2^2 = 0$  has a solution  $(x_0, x_1, x_2) = (0, 0, 1)$ . This is again impossible. Hence  $\gamma \neq \delta$ . It follows that  $\Delta$  intersects  $\Gamma$  at a point other than  $\mathbf{p}_4$  and we are in case (1).

Suppose  $\Gamma \not\subset (x_0 = x_1 = 0)$ . Then, since  $\Gamma$  is contained in either  $(x_0 = 0)$  or  $(x_1 = 0)$ , we may assume  $\Gamma = (x_0 = x_2 = y + \gamma x_1^2 = 0)$  after possibly interchanging  $x_0$  with  $x_1$  and replacing  $x_2$ . As in the above argument, we have

$$\bar{F}' := F(0, x_1, 0, y, w) = w(y^2 + \alpha_2 y x_1^2 + \alpha_4 x_1^4) + \beta_4 y x_1^4 + \beta_6 x_1^6,$$

where  $\alpha_i$  and  $\beta_i$  are the coefficients of  $x_1^i$  in  $a_i$  and  $b_i$ , respectively, and

$$\bar{F}' = (y + \gamma x_1^2)(wy + \delta w x_1^2 + \varepsilon x_1^4)$$

for some  $\delta, \varepsilon \in \mathbb{C}$ . We see that  $\varepsilon \neq 0$  because otherwise  $\beta_4 = \beta_6 = 0$  and this contradicts to Condition 2.12. It follows that  $\Delta = (x_0 = x_2 = wy + \delta y x_1^2 + \varepsilon x_1^4 = 0)$  is irreducible and reduced. If  $\gamma \neq \delta$ , then  $\Delta$  intersects  $\Gamma$  at a point other than  $\mathbf{p}_4$  and we are in case (1). If  $\beta = \gamma$ , then, after replacing  $y + \gamma x_1^2$  with  $y$ , we have  $\Gamma = (x_0 = x_2 = y = 0)$  and  $\Delta = (x_0 = x_2 = wy + \varepsilon x_1^4 = 0)$ , that is, we are in case (2). This completes the proof.  $\square$

Note that  $S$  is nonsingular along  $\Gamma \setminus \{\mathbf{p}_4\}$ .

**Lemma 10.15.** *Notation as in Lemma 10.14. Then,  $(\Gamma \cdot \Delta)_S \geq \deg \Delta$ . In particular,  $\Gamma$  is not a maximal center.*

*Proof.* If we are in case (1) of Lemma 10.14, then  $(\Gamma \cdot \Delta)_S \geq 1 > \deg \Delta$ . In the following, we assume that we are in case (2). Let  $\varphi: Y' \rightarrow X'$  be the weighted blowup of  $X'$  at  $\mathfrak{p}_4$  with  $\text{wt}(x_0, x_1, x_2, y) = \frac{1}{2}(3, 1, 1, 2)$  with exceptional divisor  $E$ . Let  $\psi = \varphi|_{\tilde{S}}: \tilde{S} \rightarrow S$  and set  $E_\psi = E|_{\tilde{S}}$ . Note that  $a_2, a_4, b_6 \in (x_0, x_2)$  and  $b_4 \notin (x_0, x_2)$ . We see that  $S$  is cut out on  $X'$  by the section  $s := \lambda x_0 + \mu x_2$ , where  $\lambda, \mu \in \mathbb{C}$  are general. We have

$$E_\psi \cong (x_0 x_1 + y^2 + y \bar{a}_2 + \bar{a}_4 = x_2 = 0) \subset \mathbb{P}(3_{x_0}, 1_{x_1}, 1_{x_2}, 2_y).$$

Since  $\bar{a}_i$  is divisible by  $x_2$ , we have  $E_\psi = (x_0 x_1 + y^2 = x_2 = 0)$ . In particular,  $E_\psi$  is quasismooth in  $\mathbb{P}(3, 1, 1, 2)$ . This implies that  $\text{rank } J_{E_\psi} = 2$ , and thus  $\text{rank } J_\psi = 2$ , at every point of  $E_\psi$ . It follows that  $\tilde{S}$  has a singular point of type  $A_2$  at  $\mathfrak{q} = (1:0:0:0)$  and it is nonsingular along  $E_\psi \setminus \{\mathfrak{q}\}$ . Note that  $\tilde{\Gamma}$  intersects  $E_\psi$  at  $(0:1:0:0) \neq \mathfrak{q}$ . By considering the resolution of  $\tilde{S}$  at  $\mathfrak{q}$ , we see that  $G(S, \mathfrak{p}_4, \Gamma)$  is of type  $A_{3,1}$ . Thus we have

$$(\Gamma^2)_S = -2 + \frac{3}{4} = -\frac{5}{4}.$$

By taking the intersection number of  $T|_S = \Gamma + \Delta$  and  $\Gamma$ , we have

$$\frac{1}{2} = (\Gamma \cdot T|_S)_S = (\Gamma^2)_S + (\Gamma \cdot \Delta)_S,$$

and then we have  $(\Gamma \cdot \Delta)_S = 7/4 > \deg \Delta$ . Therefore,  $\Gamma$  is not a maximal center by Lemma 10.3.  $\square$

**Remark 10.16.** Let  $\Gamma \subset X'$  be an irreducible and reduced curve of degree  $1/2$  passing through  $\mathfrak{p}_4$ . Then, Lemma 10.15 shows that  $(\Gamma^2)_S \leq 0$  for a general member  $S \in |\mathcal{I}_\Gamma(A)|$ . This follows by considering  $1/2 = (\Gamma \cdot T|_S)_S = (\Gamma^2)_S + (\Gamma \cdot \Delta)_S$  and  $(\Gamma \cdot \Delta)_S \geq \deg \Delta = 1/2$ . This observation will be used in the next subsection.

**10.6. Curves of degree 1 on  $X' \in \mathcal{G}'_7$ .** Let  $\Gamma$  be an irreducible and reduced curve on a member  $X'$  of  $\mathcal{G}'_7$  that passes through the  $cA/2$  point but does not pass through the other singular points. We see that  $\Gamma$  is a WCI curve of type either  $(1, 2, 2)$  or  $(1, 1, 4)$ .

We claim that  $\Gamma$  cannot be of type  $(1, 2, 2)$ . If  $\Gamma$  is of type  $(1, 2, 2)$ , then  $\Gamma = (\ell = y + c_2 = d_2 = 0)$  for some  $\ell \in \mathbb{C}[x_0, x_1, x_2]$  with  $\deg \ell = 1$  and  $c_2, d_2 \in \mathbb{C}[x_0, x_1, x_2]$  since  $\Gamma$  passes through  $\mathfrak{p}_4$ . In this case  $\Gamma$  is either reducible or non-reduced. This shows that  $\Gamma$  cannot be of type  $(1, 2, 2)$ .

**Lemma 10.17.** *An irreducible and reduced curve of degree 1 on  $X'$  passing through  $\mathfrak{p}_4$  is not a maximal center.*

*Proof.* Let  $\Gamma \subset X'$  be an irreducible and reduced curve of degree 1 passing through  $\mathfrak{p}_4$ . Then, by the above argument,  $\Gamma$  is of type  $(1, 1, 4)$ . Let  $S, T \in |\mathcal{I}_\Gamma(A)|$  be a general member and let  $\ell_1, \ell_2 \in \mathbb{C}[x_0, x_1, x_2]$  be linear forms such that  $S = (\ell_1 = 0)$  and  $T = (\ell_2 = 0)$ . Then  $T|_S = \Gamma + \Delta$ , where  $\Delta = (\ell_1 = \ell_2 = d_2 = 0)$  for some  $d_2 \in \mathbb{C}[x_0, x_1, x_2, y, w]$  of degree 2. We claim that  $d_2 \notin \mathbb{C}[x_0, x_1, x_2]$ . Indeed, if  $d_2 \in \mathbb{C}[x_0, x_1, x_2]$ , then the defining polynomial  $F'$  of  $X'$  is contained in the ideal  $(\ell_1, \ell_2, d_2) \subset (x_0, x_1, x_2)$ . This is a contradiction since  $wy^2 \in F'$ . Hence, either  $w \in d_2$  or  $y \in d_2$ , and in particular  $\Delta$  is irreducible and reduced. Suppose that  $w \in d_2$ . Then  $\Delta$  intersects  $\Gamma$  at a nonsingular point so that  $(\Gamma \cdot \Delta)_S \geq 1 > 1/2 = \deg \Delta$ . Thus  $\Gamma$  is not a maximal center. Suppose that  $w \notin d_2$ . Then  $y \in d_2$  and  $\Delta$  does

not pass through  $\mathbf{p}_4$ . Since  $S$  is a general member of  $|\mathcal{I}_\Gamma(A)| = |\mathcal{I}_\Delta(A)|$ , we have  $(\Delta^2)_S \leq 0$  by Remark 10.16. Then, we compute

$$1/2 = (\Delta \cdot T|_S) = (\Gamma \cdot \Delta)_S + (\Delta^2)_S \leq (\Gamma \cdot \Delta)_S.$$

This shows that  $\Gamma$  is not a maximal center.  $\square$

**10.7. Curves of degree 1 on  $X' \in \mathcal{G}'_9$ .** Let  $X' = X'_6 \subset \mathbb{P}(1, 1, 1, 3, 1)$  be a member of  $\mathcal{G}'_9$  and  $\Gamma \subset X'$  an irreducible and reduced curve of degree 1 that passes through  $\mathbf{p}_4$  but does not pass through the other singular points. The defining polynomial of  $X'$  can be written as  $F' = w^2x_0y + w(ya_2 + a_5) + y^2 + yb_3 + b_6$ , where  $a_j, b_j \in \mathbb{C}[x_0, x_1, x_2]$ .

**Lemma 10.18.** *An irreducible and reduced curve of degree 1 on  $X'$  passing through  $\mathbf{p}_4$  is not a maximal center.*

*Proof.* We see that  $\Gamma$  is a WCI curve of type  $(1, 1, 3)$ . Let  $S, T$  be general members of the pencil  $|\mathcal{I}_\Gamma(A)|$ . We will show that  $T|_S = \Gamma + \Delta$ , where  $\Delta \neq \Gamma$  is an irreducible and reduced curve of degree 1 such that  $(\Gamma \cdot \Delta)_S \geq \deg \Delta$ .

Suppose  $\Gamma \not\subset (x_0 = 0)$ . Then, after replacing  $x_1$  and  $x_2$ , we may assume  $\Gamma \subset (x_1 = x_2 = 0)$ . We have

$$F(x_0, 0, 0, y, w) = w^2x_0y + w(\alpha_2yx_0^2 + \alpha_5x_0^5) + y^2 + \beta_3x_0^3y + \beta_6x_0^6,$$

where  $\alpha_i$  and  $\beta_i$  are coefficients of  $x_0^i$  in  $a_i$  and  $b_i$ , respectively. Since  $\Gamma \subset X'$ , we have  $F'(x_0, 0, 0, y, w) = c_3d_3$  for some  $c_3, d_3 \in \mathbb{C}[x_0, y, w]$  of degree 3. Then, by an explicit computation, we have  $\alpha_5 = \beta_6 = 0$ ,  $c_3 = y$  and  $d_3 = w^2x_0 + \alpha_2wx_0 + y + \beta_3x_0^3$ . Note that either  $\Gamma = (x_1 = x_2 = c_3 = 0)$  or  $(x_1 = x_2 = d_3 = 0)$ . In any case,  $\Delta$  is an irreducible and reduced curve of degree 1 and we see that  $S$  is nonsingular along  $\Gamma \setminus \{\mathbf{p}_4\}$  by Lemma 10.4. Since two curves  $(x_1 = x_2 = c_3 = 0)$  and  $(x_1 = x_2 = d_3 = 0)$  have a intersection point away from  $\mathbf{p}_4$ , we have  $(\Gamma \cdot \Delta)_S \geq 1 = (A \cdot \Delta)$ . This completes the proof.

Suppose  $\Gamma \subset (x_0 = 0)$ . After replacing  $x_1, x_2$ , we may assume  $\Gamma = (x_0 = x_1 = h_3 = 0)$  for some  $h_3(x_2, y, w)$ . We have

$$G := F(0, 0, x_2, y, w) = w(\alpha_2yx_2^2 + \alpha_5x_2^5) + y^2 + \beta_3yx_2^3 + \beta_6x_2^6 = 0,$$

where  $\alpha_i, \beta_i \in \mathbb{C}$ . Note that  $\alpha_i$  is the coefficient of  $x_2^i$  in  $a_i$  so that  $(\alpha_2, \alpha_5) \neq (0, 0)$  by Condition 2.12. Since  $\Gamma \subset X'$ ,  $G$  is divisible by  $h_3$  and we can write

$$G = (y + \gamma wx_2^2 + \delta x_2^3)(y + \varepsilon wx_2^2 + \zeta x_2^3),$$

where  $h_3 = y + \gamma wx_2^2 + \delta x_2^3$  and  $\gamma, \dots, \zeta \in \mathbb{C}$ . Note that  $T|_S = \Gamma + \Delta$ , where  $\Delta = (x_0 = x_1 = y + \varepsilon wx_2^2 + \zeta x_2^3)$ . By comparing the coefficients of  $w^2x_2^4$ , we have  $\gamma\varepsilon = 0$ . If  $(\gamma, \varepsilon) \neq (0, 0)$ , then  $\gamma \neq \varepsilon$  and  $\Delta$  intersects  $\Gamma$  at a nonsingular point so that  $(\Gamma \cdot \Delta)_S \geq 1 = \deg \Delta$ . Now suppose  $\gamma = 0$ . By comparing coefficients of  $wyx_2^2$  and  $wx_2^5$ , we have  $\alpha_2 = \varepsilon$  and  $\alpha_5 = \delta\varepsilon$ . This shows  $\varepsilon \neq 0$ . It follows that  $(\gamma, \varepsilon) \neq (0, 0)$  and the proof is completed.  $\square$

**10.8. Curves of degree 1 on  $X' \in \mathcal{G}'_{10}$ .** Let  $X' = X'_6 \subset \mathbb{P}(1, 1, 2, 2, 1)$  be a member of  $\mathcal{G}'_{10}$  and  $\Gamma \subset X'$  an irreducible and reduced curve of degree 1 that passes through  $\mathbf{p}_4$  but does not pass through the other singular points. The defining polynomial of  $X'$  is of the form  $w^2y_0y_1 + wf_5 + g_6$ . Note that  $y_0^3, y_1^3 \in F'$  and we assume that the coefficients of  $y_0^3$  and  $y_1^3$  in  $F'$  are both 1 after re-scaling  $y_0, y_1$ . We see that  $\Gamma$  is a WCI curve of type either  $(1, 1, 4)$  or  $(1, 2, 2)$ .

We claim that  $\Gamma$  cannot be of type  $(1, 1, 4)$ . Indeed, a curve of type  $(1, 1, 4)$  passing through  $\mathfrak{p}_4$  is contained in  $(x_0 = x_1 = 0)$  and we have

$$F'(0, 0, y_0, y_1, w) = w^2 y_0 y_1 + y_0^3 + y_1^3,$$

which is clearly irreducible. Thus,  $X'$  cannot contain a curve of type  $(1, 1, 4)$  passing through  $\mathfrak{p}_4$ .

In the following, we treat the case where  $\Gamma$  is of type  $(1, 2, 2)$ .

**Lemma 10.19.** *After replacing  $x_0$  and  $x_1$ , and interchanging  $y_0$  with  $y_1$ , we are in one of the following cases.*

- (1)  $\Gamma = (x_0 = y_0 - \beta x_1^2 = y_1 - \gamma w x_1 - \delta x_1^2 = 0)$  for some  $\beta, \gamma, \delta \in \mathbb{C}$  with  $\gamma \neq 0$ .
- (2)  $\Gamma = (x_0 = y_0 - \beta x_1^2 = y_1 = 0)$  for some non-zero  $\beta \in \mathbb{C}$ . In this case, if  $f_5(x_0, x_1, 0, 0)$  is divisible by  $x_0^2$ , then  $\tau - \sigma_0 \sigma_1 \neq 0$ , where  $\sigma_i$  and  $\tau$  are the coefficients of  $w y_i x_1^3$  and  $x_1^6$  in  $F'$ , respectively.

*Proof.* After replacing  $x_0, x_1$ , we can write

$$\Gamma = (x_0 = y_0 - \alpha w x_1 - \beta x_1^2 = y_1 - \gamma w x_1 - \delta x_1^2 = 0)$$

for some  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ . It follows that

$$G_{\alpha, \beta, \gamma, \delta} := F(0, x_1, \alpha w x_1 + \beta x_1^2, \gamma w x_1 + \delta x_1^2) = 0$$

as a polynomial. The coefficient of  $w^4 x_1^2$  in  $G_{\alpha, \beta, \gamma, \delta}$  is  $\alpha \gamma$ . Hence  $\alpha \gamma = 0$ . After interchanging  $y_0$  and  $y_1$  if necessary, we may assume  $\alpha = 0$ . If  $\gamma \neq 0$ , then we are in case (1). Suppose  $\alpha = \gamma = 0$ . Then the coefficient of  $w^2 x_1^4$  in  $G_{\alpha, \beta, \gamma, \delta}$  is  $\beta \gamma$ . Hence  $\beta \gamma = 0$ . After interchanging  $y_0$  and  $y_1$ , we may assume  $\delta = 0$ . Hence  $\Gamma = (x_0 = y_0 - \beta x_1^2 = y_1 = 0)$ . If further  $\beta = 0$ , then  $F(0, x_1, 0, 0, w) = w f_5(0, x_1, 0, 0) + g_6(0, x_1, 0, 0) = 0$  as a polynomial, which implies that  $f_5(x_0, x_1, 0, 0)$  and  $g_6(x_0, x_1, 0, 0)$  share a common component  $x_0$ . This is impossible by Condition 2.12. Hence  $\beta \neq 0$ . It remains to show that  $\tau - \sigma_0 \sigma_1 \neq 0$  if  $f_5(x_0, x_1, 0, 0)$  is divisible by  $x_0^2$ . Let  $X \in \mathcal{G}_{10}$  be the birational counterpart of  $X'$  which is defined in  $\mathbb{P}(1, 1, 2, 2, 3, 3)$ , by

$$\begin{aligned} F_1 &= z_1 y_1 + z_0 y_0 + f_5(x_0, x_1, y_0, y_1), \\ F_2 &= z_1 z_0 - g_6(x_0, x_1, y_0, y_1). \end{aligned}$$

Now we assume that  $f_5(x_0, x_1, 0, 0)$  is divisible by  $x_0^2$  and  $\tau - \sigma_0 \sigma_1 = 0$ . Then  $X$  contains  $\mathfrak{p} := (0 : 1 : 0 : 0 : -\sigma_0 : -\sigma_1)$  and every partial derivative of  $F_1$  vanishes at  $\mathfrak{p}$ . This is a contradiction since  $X$  is quasismooth. Therefore  $\tau - \sigma_0 \sigma_1 \neq 0$  if  $f_5(x_0, x_1, 0, 0)$  is divisible by  $x_0^2$  and the proof is completed.  $\square$

**Lemma 10.20.** *No curve of type  $(1, 2, 2)$  on  $X' \in \mathcal{G}'_{10}$  passing through  $\mathfrak{p}_4$  is a maximal center.*

*Proof.* Let  $S \in |\mathcal{I}_\Gamma(2A)|$  be a general member and let  $\varphi: Y' \rightarrow X'$  be the weighted blowup of  $X'$  at  $\mathfrak{p}_4$  with  $\text{wt}(x_0, x_1, y_0, y_1) = (1, 1, 3, 2)$  with exceptional divisor  $E$ . We set  $\psi = \varphi|_{\tilde{S}}: \tilde{S} \rightarrow S$  and  $E_\psi = E|_{\tilde{S}}$ . We see that  $S$  is nonsingular along  $\Gamma \setminus \{\mathfrak{p}_4\}$  by [14, Lemma 2.5].

Suppose that  $\Gamma$  is as in (1) of Lemma 10.19. Then,  $S$  is cut out by the section

$$s := x_0(\lambda_0 x_0 + \lambda_1 x_1 + \lambda w) + \mu(y_0 - \beta x_1^2) + \nu(y_1 - \gamma w x_1 - \delta x_1^2),$$

where  $\lambda_i, \lambda, \mu, \nu \in \mathbb{C}$  are general. Note that  $K_{\tilde{S}} = \psi^* K_S$ . We have

$$E_\psi = (y_0 y_1 + \bar{f}_5 = \lambda x_0 - \nu \gamma x_1 = 0) \subset \mathbb{P}(1, 1, 2, 3)$$

and

$$J_\psi = \begin{pmatrix} \frac{\partial \bar{f}_5}{\partial x_0} & \frac{\partial \bar{f}_5}{\partial x_1} & y_1 + \frac{\partial \bar{f}_5}{\partial y_0} & y_0 & H \\ \lambda & -\nu\gamma & 0 & 0 & x_0(\lambda_0 x_0 + \lambda_1 x_1) + \mu(y_0 - \beta x_1^2) - \mu\delta x_1^2 \end{pmatrix}.$$

Since  $f_5(x_0, x_1, 0, 0) \neq 0$  as a polynomial and  $\lambda, \nu$  are general, we may assume that  $f_5(x_0, x_1, 0, 0)$  is not divisible by  $\lambda x_0 - \nu\gamma x_1$ . Then, we have

$$\left( y_1 + \frac{\partial \bar{f}_6}{\partial y_0} = y_0 = 0 \right) \cap E_\psi = \emptyset,$$

which implies that  $J_\psi$  is of rank 2 at every point of  $E_\psi$ . Thus, singular points of  $\tilde{S}$  consist of two points  $\mathbf{q}_1 := (0:0:1:0)$  and  $\mathbf{q}_2 := (0:0:0:1)$  that are of type  $A_1$  and  $A_2$ , respectively. Let  $\mathbb{A}^4$  be the orbifold chart of  $Y'$  with affine coordinates  $\tilde{x}_0, \tilde{x}_1, \tilde{y}_0, \tilde{y}_1$  such that  $x_i = \tilde{x}_i \tilde{y}_1$ ,  $y_0 = \tilde{y}_0 \tilde{y}_1^2$  and  $y_1 = \tilde{y}_0^3$ . We see that  $\mathbb{Z}_3$  acts on  $\mathbb{A}^4$  as  $\frac{1}{3}(1\tilde{x}_0, 1\tilde{x}_1, 2\tilde{y}_0, 2\tilde{y}_1)$ , and the quotient  $U := \mathbb{A}^4/\mathbb{Z}_3$  is an open subset of  $Y'$  whose origin is  $\mathbf{q}_2$ . We see that  $\tilde{S}$  is defined by  $\tilde{y}_0 + \dots = \lambda\tilde{x}_0 - \nu\gamma\tilde{x}_1 + \dots = 0$  on  $U$ . By eliminating  $\tilde{y}_0$  and  $\tilde{x}_0$ , the germ  $(\tilde{S}, \mathbf{q}_2)$  is analytically isomorphic to  $(\mathbb{A}_{\tilde{x}_1, \tilde{y}_1}^2/\mathbb{Z}_3, o)$ . Under the above isomorphism,  $E_\psi$  and  $\tilde{\Gamma}$  corresponds to  $(\tilde{y}_1 = 0)$  and  $(\tilde{x}_1 = 0)$ , respectively. Let  $\hat{S} \rightarrow \tilde{S}$  be the weighted blowup of  $\tilde{S}$  at  $\mathbf{q}_2$  with  $\text{wt}(\tilde{x}_0, \tilde{y}_1) = \frac{1}{3}(1, 2)$  and denote by  $F \cong \mathbb{P}(1, 2)$  its exceptional divisor. We see that  $\hat{S}$  has a singular point  $\hat{\mathbf{q}}$  of type  $A_1$  along  $F$  and it is nonsingular along  $F \setminus \{\hat{\mathbf{q}}\}$ . Let  $\hat{E}_\psi$  and  $\hat{\Gamma}$  be the proper transforms of  $E_\psi$  and  $\tilde{\Gamma}$  on  $\hat{S}$ , respectively. Then,  $\hat{E}_\psi$  intersects  $F$  at a nonsingular point and  $\hat{\Gamma}$  intersects  $F$  at  $\hat{\mathbf{q}}$ . Thus, by considering the blowup of  $\hat{S}$  at  $A_1$  singular points  $\hat{\mathbf{q}}$  and  $\mathbf{q}_1$ , we see that  $G(S, \mathbf{p}_4, \Gamma)$  is of type  $A_{4,1}$ .

Suppose that  $\Gamma$  is as in (2) of Lemma 10.19. Then,  $S$  is cut out by the section

$$s := x_0(\lambda_0 x_0 + \lambda_1 x_1 + \lambda w) + \mu(y_0 - \beta x_1^2) + \nu y_1,$$

where  $\lambda_i, \lambda, \mu, \nu \in \mathbb{C}$  are general. Note that  $K_{\tilde{S}} = \psi^* K_S$ . We have an isomorphism

$$E_\psi \cong (x_0 = y_0 y_1 + \bar{f}_5 = 0) \subset \mathbb{P}(1_{x_0}, 1_{x_1}, 2_{y_0}, 3_{y_1})$$

and

$$J_\psi = \begin{pmatrix} \frac{\partial \bar{f}_5}{\partial x_0} & \frac{\partial \bar{f}_5}{\partial x_1} & y_1 + \frac{\partial \bar{f}_5}{\partial y_0} & y_0 & H \\ \lambda & 0 & 0 & 0 & x_0(\lambda_0 x_0 + \lambda_1 x_1) + \mu(y_0 - \beta x_1^2) \end{pmatrix}.$$

Note that  $\tilde{\Gamma}$  intersects  $E_\psi$  at  $\mathbf{q}_0 := (0:1:\beta:0)$ . We define

$$\Sigma := \left( \frac{\partial \bar{f}_5}{\partial x_1} = y_1 + \frac{\partial \bar{f}_5}{\partial y_0} = y_0 = 0 \right) \cap E_\psi.$$

If  $f_5(x_0, x_1, 0, 0)$  is not divisible by  $x_0$ , then  $\Sigma = \emptyset$  and thus singular points of  $\tilde{S}$  consists of  $\mathbf{q}_1$  and  $\mathbf{q}_2$  that are of type  $A_1$  and  $A_2$ , respectively. In this case, by considering successive blowups of  $\tilde{S}$  at its singular points, we see that  $G(S, \mathbf{p}_4)$  is of type  $A_{4,2}$ . Suppose that  $f_5(x_0, x_1, 0, 0)$  is divisible by  $x_0$ . Then, we can write  $\bar{f}_5 = y_0(y_0 a_1 + a_3)$ , where  $a_i \in \mathbb{C}[x_0, x_1]$  and thus  $E_\psi = E_1 + E_2$ , where  $E_1 = (x_0 = y_0 = 0)$  and  $E_2 = (x_0 = y_1 + y_0 a_1 + a_3 = 0)$ . Note that  $\Sigma = \{\mathbf{q}_3\}$ , where  $\mathbf{q}_3 = (0:1:0:-\sigma_0)$  is the intersection point  $E_1 \cap E_2$ . We have

$$J_\psi(\mathbf{q}_3) = \begin{pmatrix} \rho & 0 & 0 & 0 & -\sigma_0 \sigma_1 + \tau \\ \lambda & 0 & 0 & 0 & -\mu\beta \end{pmatrix},$$



where  $\rho$  is the coefficient of  $x_0x_1^4$  in  $f_5(x_0, x_1, 0, 0)$  and  $\sigma_i, \tau$  are as in Lemma 10.19. We claim that  $\text{rank } J_\psi(\mathbf{q}_3) = 2$ . Since  $\beta \neq 0$  and  $\lambda, \mu$  are general,  $\text{rank } J_\psi(\mathbf{q}_3) < 2$  if and only if  $\rho = \tau - \sigma_0\sigma_1 = 0$ . Note that  $\rho = 0$  if and only if  $f_5(x_0, x_1, 0, 0)$  is divisible by  $x_0^2$ . Hence, the case  $\rho = \tau - \sigma_0\sigma_1 = 0$  does not happen and we have  $\text{rank } J_\psi(\mathbf{q}_3) = 2$ . It follows that  $\tilde{S}$  is nonsingular at  $\mathbf{q}_3$  and thus singular points of  $\tilde{S}$  consist of two points  $\mathbf{q}_1$  and  $\mathbf{q}_2$  that are of type  $A_1$  and  $A_2$ . Note that  $\tilde{\Gamma}$  intersects  $E_1$  at a point other than  $\mathbf{q}_1$  and does not intersect  $E_2$ . This shows that,  $G(S, \mathbf{p}_4)$  is of type  $A_{5,2}$ .

By the above argument, the type of  $G(S, \mathbf{p}_4)$  is one of  $A_{4,1}$ ,  $A_{4,2}$  and  $A_{5,2}$ . By Lemma 10.7, we have

$$(\Gamma^2)_S \leq -2 - (K_S \cdot \Gamma) + \frac{4}{3} = -\frac{5}{3}.$$

Finally, we have

$$(A^2 \cdot S) - 2(A \cdot \Gamma) + (\Gamma^2)_S \leq 3 - 2 - \frac{5}{3} < 0.$$

This shows that  $\Gamma$  is not a maximal center.  $\square$

**10.9. Curves of degree 1 on  $X' \in \mathcal{G}'_{16}$ .** Let  $X' = X'_7 \subset \mathbb{P}(1, 1, 2, 3, 1)$  be a member of  $\mathcal{G}'_{16}$  and  $\Gamma \subset X'$  an irreducible and reduced curve of degree 1 that passes through  $\mathbf{p}_4$  but does not pass through the other singular points. We see that  $\Gamma$  is a WCI curve of type either  $(1, 1, 6)$  or  $(1, 2, 3)$ . Let  $F' = w^2yz + wf_6 + g_7$  be the defining polynomial of  $X'$ . Since  $X' \in \mathcal{G}'_{16}$ , we have  $z^2, y^3 \in f_6$  and  $zy^2 \in g_7$ . After rescaling  $y, z, w$ , we assume that the coefficient of  $z^2$  and  $y^3$  in  $f_6$  are both 1. Moreover, we assume that there is no monomial divisible by  $y^3$  or  $z^2$  in  $g_7$  after replacing  $y$  and  $z$ .

We claim that  $\Gamma$  cannot be of type  $(1, 1, 6)$ . Indeed, if  $\Gamma$  of type  $(1, 1, 6)$  passing through  $\mathbf{p}_4$ , then  $\Gamma = (x_0 = x_1 = h_6 = 0)$ , where  $h_6$  is a component of

$$F'(0, 0, y, z, w) = w^2yz + w(z^2 + y^3) + \alpha y^2z.$$

Note that  $\alpha \neq 0$ . This shows that  $F'(0, 0, y, z, w)$  is irreducible so that  $h_6$  cannot be its component. This is a contradiction.

In the following, we treat curves of type  $(1, 2, 3)$ .

**Lemma 10.21.** *No curve of type  $(1, 2, 3)$  on  $X'$  is a maximal center.*

*Proof.* We can write

$$\Gamma = (x_0 = y - \alpha wx_1 - \beta x_1^2 = z + \gamma w^2x_1 + \delta wx_1^2 + \varepsilon x_1^3 = 0)$$

for some  $\alpha, \dots, \varepsilon \in \mathbb{C}$ . Let  $S \in |\mathcal{I}_\Gamma(2A)|$  be a general member, which is nonsingular along  $\Gamma \setminus \{\mathbf{p}_4\}$  by [14, Lemma 2.5], and set  $T := (x_0 = 0)_{X'}$ . We have

$$\bar{F}'_{\alpha, \beta} := F'(0, x_1, \alpha wx_1 + \beta x_1^2, z, w) = \alpha^3 w^4 x_1^3 + \alpha w^3 z x_1 + wz^2 + \dots,$$

where the omitted part is a linear combination of monomials  $w^3 x_1^4$  and  $\{w^i z^j x_1^k \mid i + 3j + k = 7, i \leq 2, j \leq 1\}$ . Since  $\Gamma \subset X'$ , we have

$$\bar{F}'_{\alpha, \beta} = (z + \gamma w^2 x_1 + \delta wx_1^2 + \varepsilon x_1^3)h_4(x_1, z, w),$$

for some  $h_4$ . We can write down  $h_4$  as  $h_4 = wz + \lambda_1 w^3 x_1 + \lambda_2 w^2 x_1^2 + \lambda_3 w x_1^3 + \lambda_4 x_1^4$  for some  $\lambda_1, \dots, \lambda_4 \in \mathbb{C}$ . We set  $\Delta := (x_0 = y - \alpha wx_1 - \beta x_1^2 = h_4 = 0) \subset X'$ . Then  $T|_S = \Gamma + \Delta$  and  $\deg \Delta = 4/3$ . By comparing the coefficients of  $w^5 x_1^2$ ,  $w^4 x_1^3$  and  $w^3 z x_1$ , we have

$$(12) \quad \gamma \lambda_1 = 0, \gamma \lambda_2 + \delta \lambda_1 = \alpha^2, \gamma + \lambda_1 = \alpha.$$

Suppose  $(\gamma, \lambda_1) \neq (0, 0)$ . Note that  $\Delta$  intersects  $\Gamma$  at 2 points if  $\lambda_1 - \gamma \neq 0$ . Since  $\gamma\lambda_1 = 0$  and  $(\gamma, \lambda_1) \neq (0, 0)$ , we have  $\lambda_1 - \gamma \neq 0$  and hence  $(\Gamma \cdot \Delta)_S \geq 2 > \deg \Delta$ . Thus,  $\Gamma$  is not a maximal center if  $\Delta$  is irreducible. Suppose that  $\Delta$  is reducible, that is,  $\lambda_4 = 0$ . Then  $\Delta = \Delta_1 + \Delta_2$ , where  $\Delta_1 = (x_0 = y - \alpha wx_1 - \beta x_1^2 = z + \lambda_1 w^2 x_1 + \lambda_2 w x_1^2 + \lambda_3 x_1^3 = 0)$  and  $\Delta_2 = (x_0 = y - \alpha wx_1 - \beta x_1^2 = w = 0)$  are irreducible and reduced curves of degree 1 and  $1/3$ , respectively. We see that both  $\Delta_1$  and  $\Delta_2$  intersect  $\Gamma$  at a point other than  $\mathbf{p}_4$ , which implies  $(\Gamma \cdot \Delta_i) \geq 1 \geq \deg \Delta_i$  for  $i = 1, 2$ . Thus,  $\Gamma$  is not a maximal center.

Suppose  $\gamma = \lambda_1 = 0$ . Then, by the equations (12), we have  $\alpha = 0$ . In this case, we have

$$\bar{F}'_{0,\beta} = (z - \delta w x_1^2 - \varepsilon x_1^3)(wz + \lambda_2 w^2 x_1^2 + \lambda_3 w x_1^3 + \lambda_4 x_1^4).$$

By comparing the coefficients of  $w^3 x_1^4$ ,  $w^2 z x_1^2$  and  $w^2 x_1^5$ , we have

$$\delta \lambda_2 = 0, \lambda_2 = \beta, \varepsilon \lambda_2 + \delta \lambda_3 = 0.$$

If  $\lambda_2 \neq 0$  or  $\lambda_2 = 0$  and  $\delta \neq 0$ , then, by the same argument as above, either  $\Delta$  is irreducible and  $(\Gamma \cdot \Delta)_S \geq 2 > \deg \Delta$  or  $\Delta = \Delta_1 + \Delta_2$  splits as a sum of irreducible and reduced curves of degree 1 and  $1/3$  such that  $(\Gamma \cdot \Delta)_S \geq 1 \geq \deg \Delta_i$  for  $i = 1, 2$ . Thus,  $\Gamma$  is not a maximal center.

In the following, we assume  $\lambda_2 = \delta = 0$ . Note that  $\beta = 0$ . Let  $S' \in |\mathcal{I}_\Gamma(3A)|$  be a general member, which is nonsingular along  $\Gamma \setminus \{\mathbf{p}_4\}$  by [14, Lemma 2.5]. We compute  $(\Gamma^2)_{S'}$ . Let  $\varphi: Y' \rightarrow X'$  be the weighted blowup at  $\mathbf{p}_4$  with  $\text{wt}(x_0, x_1, y, z) = (1, 1, 3, 3)$  and with exceptional divisor  $E$ . We set  $\psi = \varphi|_{\tilde{S}'}: \tilde{S}' \rightarrow S'$  and  $E_\psi = E|_{\tilde{S}'}$ . Let  $G$  and  $H$  be the  $\varphi$ -weight = 6 and 7 parts of  $\bar{F}(x_0, x_1, y, z, 1)$ , respectively. We write  $f_6(x_0, x_1, 0, z) = z^2 + za_3 + a_6$ , where  $a_i \in \mathbb{C}[x_0, x_1]$ . Note that  $G = yz + z^2 + za_3 + a_6$ . The surface  $S'$  is cut out by the section

$$s := d_2 x_0 + e_1 y + \mu(z - \varepsilon x_1^3),$$

where  $d_2 = d_2(x_0, x_1, y, w)$ ,  $e_1 = e_1(x_0, x_1, w)$  and  $\mu \in \mathbb{C}$ . Note that  $K_{\tilde{S}'} = \psi^* K_{S'}$ . Let  $\lambda \in \mathbb{C}$  be the coefficient of  $w^2$  in  $d_2$  so that  $\lambda x_0$  is the  $\varphi$ -weight = 1 part of  $s(x_0, x_1, y, z, 1)$ . Note that  $a_6 = f_6(x_0, x_1, 0, 0)$  is not divisible by  $x_0$  because otherwise the system of equations  $y = f_6 = g_7 = f_6(x_0, x_1, 0, 0) = 0$  has a solution  $(x_0, x_1, y, z) = (0, 1, 0, \varepsilon)$  and this is impossible by Condition 2.12. It follows that

$$E_\psi \cong (yz + z^2 + za_3 + a_6 = x_0 = 0) \subset \mathbb{P}(1, 1, 3, 3)$$

is irreducible and reduced. We have

$$J_\psi = \begin{pmatrix} \frac{\partial G}{\partial x_0} & \frac{\partial G}{\partial x_1} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} & H \\ \lambda & 0 & 0 & 0 & t \end{pmatrix},$$

where  $t = t(x_0, x_1, y, z)$  is the  $\varphi$ -weight = 2 part of  $s(x_0, x_1, y, z, 1)$ . Since  $x_0 \nmid a_6$ , we have

$$\left( \frac{\partial G}{\partial y} = \frac{\partial G}{\partial z} = 0 \right) \cap E_\psi = (z = y + a_3 = a_6 = x_0 = 0) = \emptyset.$$

It follows that the singular points of  $\tilde{S}'$  along  $E_\psi$  consists of two  $A_2$  points  $\mathbf{q}_1 := (0:0:1:0)$  and  $\mathbf{q}_2 := (0:0:1:-1)$ . Moreover,  $\tilde{\Gamma}$  intersects  $E_\psi$  at  $(0:1:0:\varepsilon) \neq \mathbf{q}_1, \mathbf{q}_2$ . By considering successive blowups at  $A_2$  points  $\mathbf{q}_1$  and  $\mathbf{q}_2$ , we see that  $G(S, \mathbf{p}_4, \Gamma)$  is of type  $A_{5,3}$ . By Lemma 10.7,  $(\Gamma^2)_S = -2 - 2 \deg \Gamma + 3/2 = -5/2$ . It follows that

$$(A^2 \cdot S) - 2(A \cdot \Gamma) + (\Gamma^2)_S = \frac{7}{2} - 2 - \frac{5}{2} < 0,$$

which implies that  $\Gamma$  is not a maximal center.  $\square$

**10.10. Curves of degree  $1/2$  on  $X' \in \mathcal{G}'_{18}$ .** Let  $X' = X'_8 \subset \mathbb{P}(1, 1, 2, 3, 2)$  be a member of  $\mathcal{G}'_{18}$  with defining polynomial  $w^2x_0z + wf_6 + g_8 = 0$  and  $\Gamma \subset X'$  an irreducible and reduced curve of degree  $1/2$  that passes through  $\mathbf{p}_4$  but does not pass through the other singular points. We see that  $\Gamma$  is a WCI curve of type either  $(1, 1, 6)$  or  $(1, 2, 3)$ .

**Lemma 10.22.** *No curve of type  $(1, 1, 6)$  on  $X'$  is a maximal center.*

*Proof.* Suppose that  $\Gamma$  is of type  $(1, 1, 6)$ . We have  $\Gamma = (x_0 = x_1 = h_6 = 0)$  for some  $h_6 \in \mathbb{C}[y, z, w]$ . Since  $z^2 \in f_6$ , we may write  $f_6(0, 0, y, z) = z^2 + \alpha y^3$  and  $h_8(0, 0, y, z) = \beta y^4 = 0$  for some  $\alpha, \beta \in \mathbb{C}$  after replacing  $w$ . Hence,

$$(x_0 = x_1 = 0)_{X'} = (x_0 = x_1 = w(z^2 + \alpha y^3) + \gamma y^4 = 0).$$

It follows that  $\gamma = 0$  and  $\Gamma = (x_0 = x_1 = z^2 + \alpha y^3 = 0)$ . Note that  $\alpha \neq 0$  since  $\Gamma$  is reduced. Let  $S$  and  $T$  be general members of the pencil  $|\mathcal{I}_\Gamma(A)|$ . We have  $T|_S = \Gamma + \Delta$ , where  $\Delta = (x_0 = x_1 = w = 0)$  is of degree 1. We see that  $S$  is nonsingular along  $\Gamma \setminus \{\mathbf{p}_4\}$  by Lemma 10.4 and  $\Gamma$  intersects  $\Delta$  in a nonsingular point. Thus  $(\Gamma \cdot \Delta) \geq 1 > (A \cdot \Delta)$ , which shows that  $\Gamma$  is not a maximal center.  $\square$

**Lemma 10.23.** *No curve of type  $(1, 2, 3)$  on  $X'$  is a maximal center.*

*Proof.* Let  $\Gamma \subset X'$  be a curve of type  $(1, 2, 3)$  passing through  $\mathbf{p}_4$ . Let  $S \in |\mathcal{I}_\Gamma(2A)|$  and  $T \in |\mathcal{I}_\Gamma(A)|$  be general members. We will show that  $T|_S = \Gamma + \Delta$ , where  $\Delta$  is an effective divisor such that  $(\Gamma \cdot \Delta_i)_S \geq \deg \Delta_i$  for every irreducible component  $\Delta_i$  of  $\Delta$ . This shows that  $\Gamma$  is not a maximal center. Note that  $S$  is nonsingular along  $\Gamma \setminus (\Gamma \cap (x_0 = x_1 = 0)) \cup \{\mathbf{p}_4\}$  by [14, Lemma 2.5]. Since  $\Gamma \cap (x_0 = x_1 = 0) = \{\mathbf{p}_4\}$  (see the descriptions of  $\Gamma$  below),  $S$  is nonsingular along  $\Gamma \setminus \{\mathbf{p}_4\}$ .

Suppose  $\Gamma \not\subset (x_0 = 0)$ . Then, after replacing  $x_1$  and  $y$ , we have  $\Gamma = (x_1 = y = z + \lambda wx_0 + \mu x_0^3 = 0)$  for some  $\lambda, \mu \in \mathbb{C}$ . We have

$$\bar{F}' := F'(x_0, 0, 0, z) = w^2x_0z + w(z^2 + \alpha_6x_0^6) + \beta_5zx_0^5 + \beta_8x_0^8,$$

where  $\alpha_6, \beta_5$  and  $\beta_8$  are the coefficients of  $wx_0^6, zx_0^5$  and  $x_0^8$  in  $F$ . Since  $\Gamma \subset X'$ , we have

$$\bar{F}' = (z - \lambda wx_0 + \lambda wx_0 + \mu x_0^3)(\gamma w^2x_0 + wz + \varepsilon wx_0^2 + \zeta zx_0^2 + \eta x_0^5)$$

for some  $\gamma, \dots, \eta \in \mathbb{C}$ . By comparing the coefficients of  $w^3x_0^2, w^2zx_0, w^2x_0^4$  and  $z^2x_0$ ,

$$\lambda\gamma = 0, \gamma + \lambda = 1, \lambda\varepsilon + \mu\gamma = 0, \zeta = 0.$$

Solving these equations, we have either  $\lambda = \mu = \zeta = 0$  and  $\gamma = 1$  or  $\gamma = \varepsilon = \zeta = 0$  and  $\lambda = 1$ . Note that  $T|_S = \Gamma + \Delta$ , where

$$\Delta = (x_1 = y = \gamma w^2x_0 + wz + \varepsilon wx_0^2 + \eta x_0^5 = 0).$$

If  $\Delta$  is irreducible, then it is reduced and it intersects  $\Gamma$  at two points other than  $\mathbf{p}_4$ , which implies  $(\Gamma \cdot \Delta)_S \geq 2 > \deg \Delta = 5/6$ . Suppose that  $\Delta$  is reducible. Then,  $\Delta = \Delta_1 + \Delta_2$ , where  $\Delta_1 = (x_1 = y = w + \zeta x_0^2 = 0)$  and  $\Delta_2 = (x_1 = y = z + \gamma wx_0 + \theta x_0^3 = 0)$  for some  $\theta \in \mathbb{C}$ . Note that  $\Delta_i \neq \Gamma$  for  $i = 1, 2$  and hence  $\Delta$  is reduced. We see that both  $\Delta_1$  and  $\Delta_2$  intersect  $\Gamma$  at a point other than  $\mathbf{p}_4$  so that  $(\Gamma \cdot \Delta_i)_S \geq 1 > \deg \Delta_i$  for  $i = 1, 2$ .

Suppose  $\Gamma \subset (x_0 = 0)$ . Then, after replacing  $y$ , we have  $\Gamma = (x_0 = y = z - \lambda wx_1 - \mu x_1^3 = 0)$  for some  $\lambda, \mu \in \mathbb{C}$ . We have

$$\bar{F} := F(0, x_1, 0, z, w) = w(z^2 + \alpha_3 zx_1^3 + \alpha_6 x_1^6) + \beta_5 zx_1^5 + \beta_8 x_1^8,$$

where  $\alpha_5, \alpha_8, \beta_5$  and  $\beta_8$  are the coefficients of  $wzx_1^3$ ,  $wx_1^6$ ,  $zx_1^5$  and  $x_1^8$ , respectively. Since  $\Gamma \subset X'$ , we have

$$\bar{F} = (z + \lambda wx_1 + \mu x_1^3)(wz + \gamma wx_1^3 + \delta zx_1^2 + \varepsilon x_1^5),$$

for some  $\gamma, \dots, \varepsilon \in \mathbb{C}$ . By comparing the coefficients of  $w^2zx_1$  and  $z^2x_1$ , we have  $\lambda = \delta = 0$ . Moreover, by comparing the other terms, we have  $\alpha_3 = \mu + \gamma$ ,  $\alpha_6 = \mu\gamma$ ,  $\beta_5 = \varepsilon$  and  $\beta_8 = \mu\varepsilon$ . If  $\gamma = \mu$ , then  $x_0 = f_6 = g_8 = \partial f_6 / \partial z = 0$  has a solution  $(x_0, x_1, y, z) = (0, 1, 0, -\mu)$ . If  $\varepsilon = 0$ , then  $\beta_5 = \beta_8 = 0$  and hence  $x_0 = b_5 = b_8 = 0$  has a solution  $(x_0, x_1, y) = (0, 1, 0)$ . These are impossible by Condition 2.12 and we have  $\mu \neq \gamma$  and  $\varepsilon \neq 0$ . We see that  $\Delta = (x_0 = y = wz + \gamma wx_1^3 + \varepsilon x_1^5 = 0)$  is irreducible and reduced since  $\varepsilon \neq 0$ , and  $\Delta$  intersects  $\Gamma$  at a point other than  $\mathbf{p}_4$  since  $\mu \neq \gamma$ . It follows that  $(\Gamma \cdot \Delta)_S \geq 1 = \deg \Delta = 5/6$ . This completes the proof.  $\square$

**10.11. Curves of degree  $1/3$  on  $X' \in \mathcal{G}'_{21}$ .** Let  $X' = X'_9 \subset \mathbb{P}(1, 1, 2, 3, 3)$  be a member of  $\mathcal{G}'_{21}$  and  $\Gamma \subset X'$  an irreducible and reduced curve of degree  $1/3$  that passes through  $\mathbf{p}_4$  but does not pass through the other singular points. Let  $F' = w^2x_0y + wf_6 + g_9$  be the defining polynomial of  $X'$ . We have  $z^2, y^3 \in f_6$ . After replacing  $w$ , we assume that  $z^3 \notin g_9$ . Then, we have  $zy^3 \in g_9$ .

We see that  $\Gamma$  is a WCI curve of type either  $(1, 1, 6)$  or  $(1, 2, 3)$ . We claim that  $\Gamma$  cannot be of type  $(1, 1, 6)$ . Indeed, if  $\Gamma$  is of type  $(1, 1, 6)$ , then  $\Gamma = (x_0 = x_1 = h_6 = 0)$  for some  $h_6 \in \mathbb{C}[y, z, w]$  of degree 6. On the other hand, we have

$$F(0, 0, y, z, w) = w(\alpha z^2 + \beta y^3) + \gamma zy^3,$$

where  $\alpha, \beta, \gamma \in \mathbb{C}$  are non-zro. Hence,  $F(0, 0, y, z, w)$  is irreducible and  $X'$  cannot contain  $\Gamma$ .

**Lemma 10.24.** *No curve of type  $(1, 2, 3)$  on  $X'$  that passes through  $\mathbf{p}_4$  is a maximal center.*

*Proof.* Let  $\Gamma$  be a curve of type  $(1, 2, 3)$  on  $X'$  passing through  $\mathbf{p}_4$ . Let  $S \in |\mathcal{I}_\Gamma(2A)|$  be a general member and  $T \in |\mathcal{I}_\Gamma(A)|$ . We will show that  $T|_S = \Gamma + \Delta$ , where  $\Delta$  is an effective divisor on  $S$  such that, for each component  $\Delta_i$  of  $\Delta$ , there exists an effective divisor  $\Xi_i$  on  $S$  such that  $(\Gamma \cdot \Xi_i)_S \geq \deg \Xi_i$  and  $(\Xi_i \cdot \Delta_j)_S \geq 0$  for  $j \neq i$ . Note that  $S$  is nonsingular along  $\Gamma \setminus \{\mathbf{p}_4\}$  by [14, Lemma 2.5] since  $\Gamma \cap (x_0 = x_1 = 0) = \{\mathbf{p}_4\}$ .

Suppose  $\Gamma \subset (x_0 = 0)$ . Then, after replacing  $z$ , we may assume  $\Gamma = (x_0 = y - \lambda x_1^2 = z = 0)$  for some  $\lambda \in \mathbb{C}$ . We have

$$F'(0, x_1, \lambda x_1^2, z, w) = w(z^2 + \alpha_3 zx_1^3 + \alpha_6 x_1^6) + \beta_6 zx_1^6 + \beta_9 x_1^9.$$

Since  $\Gamma \subset X'$ , we have  $\alpha_6 = \beta_9 = 0$  and we have

$$\Delta = (x_0 = y - \lambda x_1^2 = w(z + \alpha_3 x_1^3) + \beta_6 x_1^6 = 0).$$

If  $\alpha_3 = \lambda$ , then  $x_0 = f_6 = g_9 = \partial f_6 / \partial z = 0$  has a solution  $(x_0, x_1, y, z) = (0, 1, \lambda, 0)$ . This is impossible by Condition 2.12 and thus  $\alpha_3 \neq \lambda$ . Suppose  $\beta_6 \neq 0$ . Then  $\Delta$  is irreducible and reduced and it intersects  $\Gamma$  at a nonsingular point other than  $\mathbf{p}_4$  since  $\alpha_3 \neq \lambda$ . Thus  $(\Gamma \cdot \Delta)_S \geq 1 > \deg \Delta = 2/3$ . Suppose  $\beta_6 = 0$ . Then  $\Delta = \Delta_1 + \Delta_2$ , where  $\Delta_1 = (x_0 = y - \lambda x_1^2 = z + \alpha_3 x_1^3 = 0)$  and  $\Delta_2 = (x_0 = y = w = 0)$ . We have

$(\Gamma \cdot \Delta_2)_S = (\Delta_1 \cdot \Delta_2)_S = 1$ . Set  $\Xi_2 = \Delta_2$ . Then we have  $(\Gamma \cdot \Xi_2)_S \geq 1 > \deg \Xi_2 = 1/3$  and  $(\Xi_2 \cdot \Delta_1) \geq 0$ . By taking the intersection number of  $T|_S = \Gamma + \Delta_1 + \Delta_2$  with  $\Delta_2$ , we have  $(\Delta_2^2)_S = -5/3$  since  $(T|_S \cdot \Delta_2)_S = \deg \Delta_2 = 1/3$ . Let  $\varepsilon$  be a rational number such that  $1/2 \leq \varepsilon \leq 3/5$  and set  $\Xi_1 = \Delta_1 + \varepsilon \Delta_2$ . Then,

$$(\Gamma \cdot \Xi_1)_S - \deg \Xi_1 = (\Gamma \cdot \Delta_1) + \frac{2}{3}\varepsilon - \frac{1}{3} \geq 0,$$

and

$$(\Xi_1 \cdot \Delta_2)_S = 1 - \frac{5}{3}\varepsilon \geq 0,$$

as desired.

Suppose  $\Gamma \not\subset (x_0 = 0)$ . Then, after replacing  $x_1$  and  $z$ , we may assume  $\Gamma = (x_1 = y + \lambda x_0^2 = z = 0)$  for some  $\lambda \in \mathbb{C}$ . We have

$$F'(x_0, 0, \lambda x_1^2, z, w) = \lambda w^2 x_0^3 + w(z^2 + \alpha_3 z x_0^3 + \alpha_6 x_0^6) + \beta_6 z x_0^6 + \beta_9 x_0^9.$$

Since  $\Gamma \subset X'$ , we have  $\lambda = \alpha_6 = \beta_9 = 0$  and

$$\Delta = (x_1 = y = w(z + \alpha_3) + \beta_6 x_0^6 = 0).$$

If  $\alpha_3 = 0$ , then  $y = f_6 = g_9 = \partial f_6 / \partial z = 0$  has a solution  $(x_0, x_1, y, z) = (1, 0, 0, 0)$ . This is impossible by Condition 2.12 and thus  $\alpha_3 \neq 0$ . Since  $\Gamma \subset X'$ ,  $F'(x_0, 0, y, 0, w)$  is divisible by  $y - \eta x_0^2$ , which happens if and only if  $\eta = 0$  and both  $f_6(x_0, 0, y, 0)$  and  $g_9(x_0, 0, y, w)$  are divisible by  $y$ . Thus,  $\Gamma = (x_1 = y = z = 0)$ . Suppose  $\beta_6 \neq 0$ . Then,  $\Delta$  is irreducible and reduced and  $(\Gamma \cdot \Delta)_S \geq 1$  since  $\alpha_3 \neq 0$ . Suppose  $\beta_6 = 0$ . Then,  $\Delta = \Delta_1 + \Delta_2$ , where  $\Delta_1 = (x_1 = y = z + \alpha_3 x_0^3 = 0)$  and  $\Delta_2 = (x_1 = y = w = 0)$ . We have  $(\Gamma \cdot \Delta_2)_S = (\Gamma \cdot \Delta_2)_S = 1$ . As in the above argument,  $\Xi_1 = \Delta_2 + \varepsilon \Delta_2$ , where  $1/2 \leq \varepsilon \leq 3/5$ , and  $\Xi_2 = \Delta_2$  are the desired effective divisors. This completes the proof.  $\square$

The following is the conclusion of this section.

**Theorem 10.25.** *Let  $X'$  be a member of  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$ . Then, no curve on  $X'$  is a maximal center.*

*Proof.* This follows from Lemma 10.1 and the conclusions of Sections 10.3–10.11.  $\square$

## 11. THE BIG TABLE

The list of the families  $\mathcal{G}_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$  is given in Table 6 and we list the families  $\mathcal{G}'_i$  with  $i \in I_{cA/n}^* \cup I_{cD/3}$  below. In each family  $\mathcal{G}'_i$ , a standard defining equation is described. The monomials right after the equation is a condition imposed on the family (see Remark 2.10). The table of each family is divided into two parts: terminal quotient parts and  $\mathfrak{p}_4$  parts.

We first explain terminal quotient parts. The first column indicates the number and type of the singular points. The second column indicates how to exclude them if it is non-empty. If a set of polynomials and a divisor of the form  $bB + eE$  are given, then it is excluded in Proposition 8.1. If the inequality on  $B^3$  is given and no other information is given in the second column, then the point is excluded in one of Propositions 8.3–8.6. The third column indicates the existence of birational involution that is a Sarkisov link centered at the corresponding point (see Section 5).

We next explain  $\mathfrak{p}_4$  parts of the table. We include in this table the equation of the singularity, blowup weights of divisorial extraction. For family No.  $i$  with  $i \in I_{cA/n}^*$ ,

TABLE 6. Families  $\mathcal{G}_i$

No.	$X_{d_1, d_2} \subset \mathbb{P}(a_0, \dots, a_5)$	No.	$X_{d_1, d_2} \subset \mathbb{P}(a_0, \dots, a_5)$
6	$X_{4,5} \subset \mathbb{P}(1, 1, 1, 2, 2, 3)$	33	$X_{9,10} \subset \mathbb{P}(1, 1, 3, 4, 5, 6)$
7	$X_{4,6} \subset \mathbb{P}(1, 1, 1, 2, 3, 3)$	36	$X_{8,12} \subset \mathbb{P}(1, 1, 3, 4, 5, 7)$
9	$X_{5,6} \subset \mathbb{P}(1, 1, 1, 2, 3, 4)$	38	$X_{9,12} \subset \mathbb{P}(1, 2, 3, 4, 5, 7)$
10	$X_{5,6} \subset \mathbb{P}(1, 1, 2, 2, 3, 3)$	44	$X_{10,12} \subset \mathbb{P}(1, 2, 3, 5, 5, 7)$
16	$X_{6,7} \subset \mathbb{P}(1, 1, 2, 3, 3, 4)$	48	$X_{11,12} \subset \mathbb{P}(1, 1, 4, 5, 6, 7)$
18	$X_{6,8} \subset \mathbb{P}(1, 1, 2, 3, 3, 5)$	52	$X_{10,15} \subset \mathbb{P}(1, 2, 3, 5, 7, 8)$
21	$X_{6,9} \subset \mathbb{P}(1, 1, 2, 3, 4, 5)$	57	$X_{12,14} \subset \mathbb{P}(1, 2, 3, 5, 7, 9)$
22	$X_{7,8} \subset \mathbb{P}(1, 1, 2, 3, 4, 5)$	61	$X_{12,15} \subset \mathbb{P}(1, 1, 4, 5, 6, 11)$
26	$X_{8,9} \subset \mathbb{P}(1, 1, 3, 4, 4, 5)$	62	$X_{12,15} \subset \mathbb{P}(1, 3, 4, 5, 6, 9)$
28	$X_{8,10} \subset \mathbb{P}(1, 1, 2, 3, 5, 7)$	63	$X_{12,15} \subset \mathbb{P}(1, 3, 4, 5, 7, 8)$

we also indicates what happens for each divisorial extraction. The mark “none” indicates that the corresponding divisorial extraction is not a maximal extraction (see Section 9). The mark “ $X' \dashrightarrow X \ni \frac{1}{r}(\alpha, \beta, \gamma)$ ” (resp. “B.I.”) indicates that there is a Sarkisov link starting with the corresponding divisorial extraction that is a link to  $X$  ending with the Kawamata blowup centered at a  $\frac{1}{r}(\alpha, \beta, \gamma)$  point of  $X$  (resp. that is a birational involution) (see Section 4 and 6, respectively).

No. 6:  $X_5 \subset \mathbb{P}(1, 1, 1, 2, 1)$ ,  $(A^3) = 5/2$ .

Eq:  $w^2x_0y + wf_4 + g_5$ ,  $y^2 \in f_4$ .

$\mathbf{p}_3 = \frac{1}{2}(1, 1, 1)$		Q.I.
$\mathbf{p}_4 = cA$	$x_0y + h_4(x_1, x_2)$	wt = $(a, b, 1, 1)$
$(1, 3), (3, 1)$	$X' \dashrightarrow X \ni \frac{1}{2}(1, 1, 1)$	$(2, 2)$ $X' \dashrightarrow X \ni \frac{1}{3}(1, 1, 2)$

No. 7:  $X_6 \subset \mathbb{P}(1, 1, 1, 2, 2)$ ,  $(A^3) = 3/2$ .

Eq:  $w^2x_0x_1 + wf_4 + g_6$ .

$\mathbf{p}_3\mathbf{p}_4 = \frac{1}{2}(1, 1, 1)$		Q.I.
$\mathbf{p}_4 = cA/2$	$x_0x_1 + h_4(x_1, y)/\mathbb{Z}_2(1, 1, 1, 0)$	wt = $\frac{1}{2}(a, b, 1, 2)$
$(1, 3), (3, 1)$	$X' \dashrightarrow X \ni \frac{1}{3}(1, 1, 2)$	

No. 9:  $X_6 \subset \mathbb{P}(1, 1, 1, 3, 1)$ ,  $(A^3) = 2$ .

Eq:  $w^2x_0y + wf_5 + g_6$ .

$\mathbf{p}_4 = cA$	$x_0y + h_5(x_1, x_2)$	wt = $(a, b, 1, 1)$
$(1, 4), (4, 1)$	$X' \dashrightarrow X \ni \frac{1}{2}(1, 1, 1)$	$(2, 3), (3, 2)$ $X' \dashrightarrow X \ni \frac{1}{4}(1, 1, 3)$

No. 10:  $X_6 \subset \mathbb{P}(1, 1, 2, 2, 1)$ ,  $(A^3) = 3/2$ .

Eq:  $w^2y_0y_1 + wf_5 + g_6$ ,  $y_0^3 \in g_6$  and  $y_1^3 \in g_6$ .

$p_2p_3 = 3 \times \frac{1}{2}(1, 1, 1)$		Q.I.
$p_4 = cA$	$y_0y_1 + h_5(x_0, x_1)$	wt = $(a, b, 1, 1)$
$(1, 4), (4, 1)$	B.I.	$(2, 3), (3, 2)$
		$X' \dashrightarrow X \ni \frac{1}{3}(1, 1, 2)$

No. 16:  $X_7 \subset \mathbb{P}(1, 1, 2, 3, 1)$ ,  $(A^3) = 7/6$ .

Eq:  $w^2yz + wf_6 + g_7$ ,  $z^2, y^3 \in f_6, y^2z \in g_7$ .

$p_2 = \frac{1}{2}(1, 1, 1)$		Q.I.
$p_3 = \frac{1}{3}(1, 1, 2)$		Q.I.
$p_4 = cA$	$yz + h_6(x_0, x_1)$	wt = $(a, b, 1, 1)$
$(1, 5), (5, 1)$	none	$(2, 4), (4, 2)$
		$X' \dashrightarrow X \ni \frac{1}{3}(1, 1, 2)$
$(3, 3)$	$X' \dashrightarrow X \ni \frac{1}{4}(1, 1, 3)$	

No. 18:  $X_8 \subset \mathbb{P}(1, 1, 2, 3, 2)$ ,  $(A^3) = 2/3$ .

Eq:  $w^2x_0z + wf_6 + g_8$ ,  $z^2 \in f_6$ .

$p_2p_4 = \frac{1}{2}(1, 1, 1)$		E.I. $*z^2y$
$p_3 = \frac{1}{3}(1, 1, 2)$		Q.I.
$p_4 = cA/2$	$x_0z + h_6(x_1, y)/\mathbb{Z}_2(1, 1, 1, 0)$	wt = $\frac{1}{2}(a, b, 1, 2)$
$(1, 5), (5, 1)$	$X' \dashrightarrow X \ni \frac{1}{3}(1, 1, 1)$	$(3, 3)$
		$X' \dashrightarrow X \ni \frac{1}{5}(1, 2, 3)$

No. 21:  $X_9 \subset \mathbb{P}(1, 1, 2, 3, 3)$ ,  $(A^3) = 1/2$ .

Eq:  $w^2x_0y + wf_6 + g_9$ ,  $z^2, y^3 \in f_6$ , and  $zy^3 \in g_9$  if  $z^3 \notin g_9$ .

$p_2 = \frac{1}{2}(1, 1, 1)$	$B^3 = 0$	
$p_3p_4 = \frac{1}{3}(1, 1, 2)$		Q.I.
$p_4 = cA/3$	$x_0y + h_6(x_1, z)/\mathbb{Z}_3(1, 2, 1, 0)$	wt = $\frac{1}{3}(a, b, 1, 3)$
$(1, 5)$	$X' \dashrightarrow X \ni \frac{1}{4}(1, 1, 3)$	$(4, 2)$
		$X' \dashrightarrow X \ni \frac{1}{5}(1, 2, 3)$

No. 22:  $X_8 \subset \mathbb{P}(1, 1, 2, 4, 1)$ ,  $(A^3) = 1$ .

Eq:  $w^2yz + wf_7 + g_8$ ,  $y^4 \in g_8$ .

$p_2p_3 = 2 \times \frac{1}{2}(1, 1, 1)$		Q.I.
$p_4 = cA$	$yz + h_7(x_0, x_1)$	wt = $(a, b, 1, 1)$
$(1, 6), (6, 1)$	none	$(2, 5), (5, 2)$
		$X' \dashrightarrow X \ni \frac{1}{3}(1, 1, 2)$
$(3, 4), (4, 3)$	$X' \dashrightarrow X \ni \frac{1}{5}(1, 1, 4)$	



No. 26:  $X_9 \subset \mathbb{P}(1, 1, 3, 4, 1)$ ,  $(A^3) = 3/4$ .

Eq:  $w^2yz + wf_8 + g_9$ ,  $z^2 \in f_8$ .

$\mathbf{p}_3 = \frac{1}{4}(1, 1, 3)$			Q.I.
$\mathbf{p}_4 = cA$	$yz + h_8(x_0, x_1)$		$\text{wt} = (a, b, 1, 1)$
$(1, 7), (7, 1)$	none	$(2, 6), (6, 2)$	B.I.
$(3, 5), (5, 3)$	$X' \dashrightarrow X \ni \frac{1}{4}(1, 1, 3)$	$(4, 4)$	$X' \dashrightarrow X \ni \frac{1}{5}(1, 1, 4)$

No. 28:  $X_{10} \subset \mathbb{P}(1, 1, 2, 5, 2)$ ,  $(A^3) = 1/2$ .

Eq:  $w^2x_0z + wf_8 + g_{10}$ .

$\mathbf{p}_2\mathbf{p}_4 = \frac{1}{2}(1, 1, 1)$	$B^3 = 0, \{x_0, x_1, w'\}, B$		
$\mathbf{p}_4 = cA/2$	$x_0z + h_8(x_1, y)/\mathbb{Z}_2(1, 1, 1, 0)$		$\text{wt} = \frac{1}{2}(a, b, 1, 2)$
$(1, 7), (7, 1)$	$X' \dashrightarrow X \ni \frac{1}{3}(1, 1, 2)$	$(3, 5), (5, 3)$	$X' \dashrightarrow X \ni \frac{1}{7}(1, 2, 5)$

No. 33:  $X_{10} \subset \mathbb{P}(1, 1, 3, 5, 1)$ ,  $(A^3) = 2/3$ .

Eq:  $w^2yz + wf_9 + g_{10}, y^3 \in f_9$ .

$\mathbf{p}_2 = \frac{1}{3}(1, 1, 2)$	$B^3 > 0$		
$\mathbf{p}_4 = cA$	$yz + h_9(x_0, x_1)$		$\text{wt} = (a, b, 1, 1)$
$(1, 8), (8, 1)$	none	$(2, 7), (7, 2)$	B.I.
$(3, 6), (6, 3)$	$X' \dashrightarrow X \ni \frac{1}{4}(1, 1, 3)$	$(4, 5), (5, 4)$	$X' \dashrightarrow X \ni \frac{1}{6}(1, 1, 5)$

No. 36:  $X_{12} \subset \mathbb{P}(1, 1, 3, 4, 4)$ ,  $(A^3) = 1/4$ .

Eq:  $w^2x_0y + wf_8 + g_{12}$ .

$\mathbf{p}_3\mathbf{p}_4 = \frac{1}{4}(1, 1, 3)$			Q.I.
$\mathbf{p}_4 = cA/4$	$x_0y + h_8(x_1, z)/\mathbb{Z}_4(1, 3, 1, 0)$		$\text{wt} = \frac{1}{4}(a, b, 1, 4)$
$(1, 7)$	$X' \dashrightarrow X \ni \frac{1}{5}(1, 1, 4)$	$(5, 3)$	$X' \dashrightarrow X \ni \frac{1}{7}(1, 3, 4)$

No. 38:  $X_{12} \subset \mathbb{P}(1, 2, 3, 4, 3)$ ,  $(A^3) = 1/6$ .

Eq:  $w^2yt + wf_9 + g_{12}$ ,  $y^6 \in g_{12}$ .

$\mathbf{p}_1\mathbf{p}_3 = 3 \times \frac{1}{2}(1, 1, 1)$	$B^3 < 0, \{x, z, w\}, 3B + E$		
$\mathbf{p}_2\mathbf{p}_4 = \frac{1}{3}(1, 1, 2)$	$B^3 = 0, \{x, y, w'\}, B$		
$\mathbf{p}_4 = cA/3$	$yt + h_9(x, z)/\mathbb{Z}_3(2, 1, 1, 0)$		$\text{wt} = \frac{1}{3}(a, b, 1, 3)$
$(2, 7)$	$X' \dashrightarrow X \ni \frac{1}{5}(1, 2, 3)$	$(5, 4)$	$X' \dashrightarrow X \ni \frac{1}{7}(1, 3, 4)$
$(8, 1)$	B.I.		

No. 44:  $X_{12} \subset \mathbb{P}(1, 2, 3, 5, 2)$ ,  $(A^3) = 1/5$ .

Eq:  $w^2zt + wf_{10} + g_{12}$ ,  $t^2 \in f_{10}$ .

$\mathbf{p}_1\mathbf{p}_4 = \frac{1}{2}(1, 1, 1)$	$B^3 < 0$	
$\mathbf{p}_3 = \frac{1}{5}(1, 2, 3)$		Q.I.
$\mathbf{p}_4 = cA/2$	$zt + h_{10}(x, y)/\mathbb{Z}_2(1, 1, 1, 0)$	$\text{wt} = \frac{1}{2}(a, b, 1, 2)$
$(1, 9), (9, 1)$	none	$(3, 7), (7, 3)$
$(5, 5)$	$X' \dashrightarrow X \ni \frac{1}{7}(1, 2, 5)$	

No. 48:  $X_{12} \subset \mathbb{P}(1, 1, 4, 6, 1)$ ,  $(A^3) = 1/2$ .

Eq:  $w^2yz + wf_{11} + g_{12}$ .

$\mathbf{p}_2\mathbf{p}_3 = \frac{1}{2}(1, 1, 1)$	$B^3 = 0, \{x_0, x_1, w\}, B$	
$\mathbf{p}_4 = cA$	$yz + h_{11}(x_0, x_1)$	$\text{wt} = (a, b, 1, 1)$
$(1, 10), (10, 1)$	none	$(2, 9), (9, 2)$
$(3, 8), (8, 3)$	B.I.	$(4, 7), (7, 4)$
$(5, 6), (6, 5)$	$X' \dashrightarrow X \ni \frac{1}{7}(1, 1, 6)$	

No. 52:  $X_{15} \subset \mathbb{P}(1, 2, 3, 5, 5)$ ,  $(A^3) = 1/10$ .

Eq:  $w^2yz + wf_{10} + g_{15}$ ,  $t^2 \in f_{10}$ .

$\mathbf{p}_1 = \frac{1}{2}(1, 1, 1)$	$B^3 < 0, \{x, z, t, w\}, 5B + 2E$	
$\mathbf{p}_3\mathbf{p}_4 = \frac{1}{5}(1, 2, 3)$		Q.I.
$\mathbf{p}_4 = cA/5$	$yz + h_{10}(x, t)/\mathbb{Z}_5(2, 3, 1, 0)$	$\text{wt} = \frac{1}{5}(a, b, 1, 5)$
$(2, 8)$	$X' \dashrightarrow X \ni \frac{1}{7}(1, 2, 5)$	$(7, 3)$

No. 57:  $X_{14} \subset \mathbb{P}(1, 2, 3, 7, 2)$ ,  $(A^3) = 1/6$ .

Eq:  $w^2zt + wf_{12} + g_{14}$ ,  $z^4 \in f_{12}$ .

$\mathbf{p}_1\mathbf{p}_4 = \frac{1}{2}(1, 1, 1)$	$B^3 < 0, \{x, z, w'\}, 3B + E$	
$\mathbf{p}_2 = \frac{1}{3}(1, 1, 2)$	$B^3 = 0, \{x, y, w\}, B$	
$\mathbf{p}_4 = cA/2$	$zt + h_{12}(x, y)/\mathbb{Z}_2(1, 1, 1, 0)$	$\text{wt} = \frac{1}{2}(a, b, 1, 2)$
$(1, 11), (11, 1)$	none	$(3, 9), (9, 3)$
$(5, 7), (7, 5)$	$X' \dashrightarrow X \ni \frac{1}{9}(1, 2, 7)$	

No. 61:  $X_{15} \subset \mathbb{P}(1, 1, 5, 6, 3)$ ,  $(A^3) = 1/6$ .

Eq:  $w^4x_0^3 + w^3x_0^2f_4 + w^2x_0f_8 + wf_{12} + g_{15}$ .

$\mathbf{p}_3 = \frac{1}{6}(1, 1, 5)$		Q.I.
$\mathbf{p}_4 = cD/3$		$X' \dashrightarrow X \ni \frac{1}{11}(1, 5, 6)$

No. 62:  $X_{15} \subset \mathbb{P}(1, 3, 4, 5, 3)$ ,  $(A^3) = 1/12$ .

Eq:  $w^3y^2 + w^2yf_6 + wf_{12} + g_{15}$ .

$p_1p_4 = 3 \times \frac{1}{3}(1, 1, 2)$	$B^3 < 0$	
$p_2 = \frac{1}{4}(1, 1, 3)$	$B^3 = 0, \{x, y, w\}, B$	
$p_4 = cD/3$		$X' \dashrightarrow X \ni \frac{1}{9}(1, 4, 5)$

No. 63:  $X_{15} \subset \mathbb{P}(1, 3, 4, 5, 3)$ ,  $(A^3) = 1/12$ .

Eq:  $w^2zt + wf_{12} + g_{15}, z^3 \in f_{12}$ .

$\mathbf{p_1p_4} = \frac{1}{3}(1, 1, 2)$		$B^3 < 0$		
$\mathbf{p_2} = \frac{1}{4}(1, 1, 3)$		$B^3 = 0, \{x, y, w\}, B$		
$\mathbf{p_4} = cA/3$		$zt + h_{12}(x, y)/\mathbb{Z}_3(1, 2, 1, 0)$		$\text{wt} = \frac{1}{3}(a, b, 1, 3)$
$(1, 11)$	none		$(4, 8)$	$X' \dashrightarrow X \ni \frac{1}{7}(1, 1, 2)$
$(7, 5)$	$X' \dashrightarrow X \ni \frac{1}{5}(1, 1, 3)$		$(10, 2)$	B.I.

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